THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial Week 6

MATH3968: Differential Geometry — Semester 2, 2009

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"Lecture notes" refers to *Lecture Notes for Differential Geometry, MATH3968* by Nigel O'Brien. "do Carmo" refers to *Differential Geometry of Curves and Surfaces*, by Manfredo do Carmo.

Solutions to exercises in the class notes are posted separately; below are solutions to the remaining exercises.

Required Problems

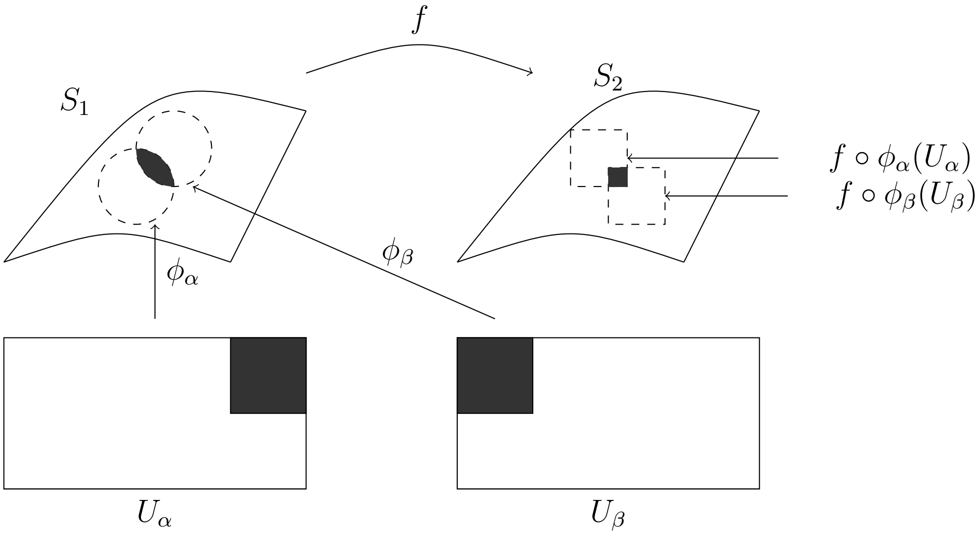
1. do Carmo §2.6 p109 q5

Solution: Let $f : S_1 \to S_2$ be a diffeomorphism.

(a) We need to show S_1 is orientable if and only if S_2 is orientable, i.e. that orientability is preserved by diffeomorphisms.
So suppose S_1 is orientable. Then there exists an atlas $\{\phi_\alpha : U_\alpha \to S_1\}$ such that

$$\det\left[d\left(\phi_\beta^{-1} \circ \phi_\alpha\right)\right] > 0$$

for all α, β with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$.



Then since f is a diffeomorphism,

$$\{f \circ \phi_\alpha : U_\alpha \to S_2\}$$

is an atlas for S_2, and

$$\begin{aligned}\det\left\{d\left[(f \circ \phi_\beta)^{-1} \circ (f \circ \phi_\alpha)\right]\right\} &= \det\left[d\left(\phi_\beta^{-1} \circ f^{-1} \circ f \circ \phi_\alpha\right)\right] \\ &= \det\left[d\left(\phi_\beta^{-1} \circ \phi_\alpha\right)\right] > 0.\end{aligned}$$

(b) Suppose now S_1 and S_2 are orientable *and* oriented. Let the orientation on S be given by an atlas $\{\phi_\alpha : U_\alpha \to S_1\}$ and let the orientation on S_2 be given by an atlas $\{\psi_a : V_a \to S_2\}$. As above, the diffeomorphism f induces an atlas

$$\{f \circ \phi_\alpha : U_\alpha \to S_2\}$$

such that all change of coordinate Jacobians have positive determinant, and hence an orientation on S_2.

The orientation given by the atlas $\{f \circ \phi_\alpha\}$ may not be the same as that given by the atlas $\{\psi_a\}$. Consider as a counterexample the unit 2-sphere

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

with the *antipodal map*

$$\begin{array}{rcl} A : \mathbb{S}^2 & \to & \mathbb{S}^2 \\ (x, y, z) & \mapsto & (-x, -y, -z), \end{array}$$

which maps every point on the sphere to its antipode.

Let N by the smooth field of unit normal vectors on $\mathbb{S}^2$ given by an atlas

$$\{\phi_\alpha : U_\alpha \to \mathbb{S}^2, \quad (u_\alpha^1, u_\alpha^2) \mapsto \phi_\alpha(u_\alpha^1, u_\alpha^2)\}.$$

Take $p \in \mathbb{S}^2$ and a local parametrisation $\phi_\alpha : U_\alpha \to \mathbb{S}^2$ near p. Then

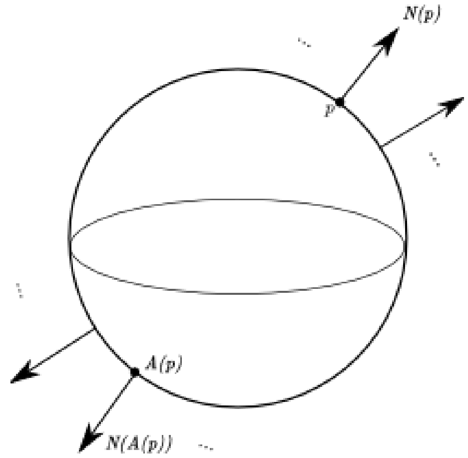
$$N(p) = \left. \frac{\frac{\partial \phi_\alpha}{\partial u_\alpha^1} \times \frac{\partial \phi_\alpha}{\partial u_\alpha^2}}{\left| \frac{\partial \phi_\alpha}{\partial u_\alpha^1} \times \frac{\partial \phi_\alpha}{\partial u_\alpha^2} \right|} \right|_{\phi_\alpha^{-1}(p)}.$$

However the unit normal vector field induced by the atlas $\{A \circ \phi_\alpha : U_\alpha \to \mathbb{S}^2\}$ is

$$\{-\phi_\alpha : U_\alpha \to \mathbb{S}^2\}.$$

Then $-\phi_\alpha$ is a local parametrisation $U_\alpha \to \mathbb{S}^2$ near $A(p)$, the antipode of p, and

$$\tilde{N}(A(p)) = \left. \frac{\frac{\partial(-\phi_\alpha)}{\partial u_\alpha^1} \times \frac{\partial(-\phi_\alpha)}{\partial u_\alpha^2}}{\left| \frac{\partial(-\phi_\alpha)}{\partial u_\alpha^1} \times \frac{\partial(-\phi_\alpha)}{\partial u_\alpha^2} \right|} \right|_{-\phi_\alpha^{-1}(A(p))} = N(p) = -N(A(p)).$$



2. Lecture Notes, Exercise Set 6, Q3

3. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces.

(a) paraboloid of revolution $z = x^2 + y^2$;

(b) hyperboloid of revolution $x^2 + y^2 - z^2 = 1$;

(c) catenoid $x^2 + y^2 = \cosh^2 z$.

c.f. do Carmo §3.2 p151 Q8

Solution:

(a) There is an obvious parametrization $\phi(x, y) = (x, y, x^2 + y^2)$ for $(x, y) \in \mathbb{R}^2$. Hence $\mathbf{E}_1 = (1, 0, 2x)$ and $\mathbf{E}_2 = (0, 1, 2y)$, so $\mathbf{E}_1 \times \mathbf{E}_2 = (-2x, -2y, 1)$. Thus

$$\mathbf{N}(\phi(x, y)) = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}(-2x, -2y, 1).$$

The image of $\mathbf{N}$ is the upper hemisphere not including the equator, since the coefficient of the z-component is > 0. (It is easy to see that all such points are realized.)

(b) In this case there is no obvious parametrization.

It is better to use the fact that the surface is defined by the single equation $f(x, y, z) = 0$, where $f(x, y, z) = x^2 + y^2 - z^2 - 1$. Hence the normal direction is given by $\nabla(f) = (2x, 2y, -2z)$, which has length $2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{1 + 2z^2}$. Hence

$$\mathbf{N} = \frac{1}{\sqrt{1 + 2z^2}}(x, y, -z)$$

Since $|\frac{z}{\sqrt{1+2z^2}}| < \frac{1}{\sqrt{2}}$ the image of N is an open "tropical zone" around the equator. (It is again easy to see that all such points are realized.)

(c) Here we use $f(x, y, z) = x^2 + y^2 - \cosh^2 z$. Then

$$|\nabla(f)| = 2\sqrt{x^2 + y^2 + \cosh^2 z \sinh^2 z} = 2\cosh z\sqrt{(1 + \sinh^2 z} = 2\cosh^2 z,$$

and so

$$\mathbf{N} = \frac{x}{\cosh^2 z}\mathbf{i} + \frac{y}{\cosh^2 z}\mathbf{j} - \tanh z\mathbf{k}).$$

Since $|\tanh z| < 1$ the image of N does not contain the poles $(0, 0, \pm 1)$.

4. Show that the sum of the normal curvatures for any pair of orthogonal directions at a given point $p \in S$ is $2H(p)$.

Solution: Let v_1 and v_2 be unit vectors in the principal directions.

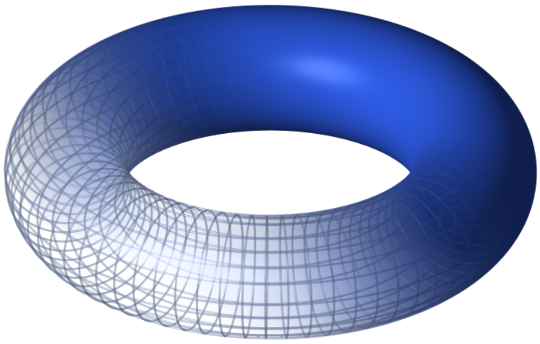
If v is any other unit vector in $T_P(S)$ and $\langle v, v_1\rangle_P = \cos\theta$ then $v = cv_1 + sv_2$, where $c = \cos\theta$ and $s = \sin\theta$. Hence $k_n(v) = II_P(v) = k_1c^2 + k_2s^2$.

If $w \perp v$ is another unit vector in $T_P(S)$ then $w = \pm sv_1 \pm cv_2$.

Hence $k_n(v) + k_n(w) = k_1(c^2 + s^2) + k_2(s^2 + c^2) = k_1 + k_2 = 2H$.

5. do Carmo §3.2 p151 Q16

Solution: A *meridian* for the torus is the circle made by intersecting the torus with a plane through the origin, parallel to the z-axis, i.e. the small circles in the picture below:



Consider the symmetry of the torus through any meridian. Take a plane through the origin parallel to the z-axis, whose cross-section with the torus is a meridian.

A principal direction which was not preserved under reflection in such a plane would not be unique; hence the only principal directions can be along such a plane, and must also be on the surface, so must lie on the meridian - meaning the meridians are lines of curvature.

To see this argument expressed mathematically, let α be the curve on the torus Σ which is a meridian (see above). We want $N' = dN_p(\alpha') = k\alpha'$. Now we know $N' \in T_p\Sigma$, by definition, and since α lies in a plane, say P, $N' \in P$ and so

$$N' \in T_p\Sigma \cap P = \mathbb{R}\alpha',$$

the line spanned by α'.

Note that the cross-sections formed by intersecting horizontal planes with the torus will also be lines of curvature.

6. do Carmo §3.2 p 151 Q13

Solution: See do Carmo *Hints and Solutions* p482

Recommended Problems

7. Lecture Notes, Exercise Set 6, Q5

8. Show that if a surface S is tangent to a plane along a curve $\alpha \subset S$ then $\det(dN_p) = 0$ at all points p on the curve.

c.f. do Carmo §3.2 p151 Q2

Solution: Suppose that $\alpha : I \to S$. Since $N(\alpha(t))$ is constant for $t \in I$ differentiation gives $dN(\alpha(t))(\alpha'(t)) = 0$ for all t. Since $\alpha'(t) \neq 0$ it follows that $\det(dN_p) = 0$ for all p on the curve.

9. Assume that the principal curvatures of a surface S have absolute value ≤ 1.

Is it true that the curvature k of a curve on S is also bounded by 1?

c.f. do Carmo §3.2 p151 Q4

Solution: No. Let S be a plane. Then all the principle curvatures of S are 0. However small circles on S have large curvature.