THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 1 (Week 2)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Outcomes

After completing this tutorial you should

(1) understand the basic rules of set theory as needed for measure theory.

(2) be able to work with inverse images of functions.

(3) use the rules of set theory and inverse images in relation to σ-algebras.

(4) be able to work with the definition of the Lebesgue outer measure.

For your reference we collect some facts on set theory. We always look at a base set X and subsets thereof. Here are some precise definition:

- $A \subseteq X$ is a subset of X if $x \in A$ implies $x \in X$. The collection of all subsets of X is called the *power set* of X. We denote the power set of X by $\mathcal{P}(X)$.
- If $A, B \subseteq X$ we write $A \subseteq B$ if $x \in B$ for every $x \in A$ and $A = B$ if $A \subseteq B$ and $B \subseteq A$.
- Let I be an arbitrary index set, for instance $I = \{1, 2, \dots, n\}$ finite, $I = \mathbb{N}$ countable or $I = \mathbb{R}$ uncountable. There is no restriction on the cardinality of I. Suppose that for every $i \in I$ there is a set $A_i \subseteq X$. We define the *union*
$$\bigcup_{i \in I} A_i := \{x \in X : x \in A_i \text{ for some } i \in I\}$$
and the *intersection*
$$\bigcap_{i \in I} A_i := \{x \in X : x \in A_i \text{ for all } i \in I\}.$$
- We define the *complement* A^c of $A \subseteq X$ by
$$A^c := X \setminus A := \{x \in X : x \notin A\}.$$
For $X \setminus A$ we also say "X minus A".

As in algebra, there are rules on how to do computations with sets. We state the most basic ones below. We assume that $A_i \subseteq X$ for all $i \in I$ and $B \subseteq X$, where I is an arbitrary index set.

Distributive laws:

$$\Bigl(\bigcap_{i \in I} A_i\Bigr) \cup B = \bigcap_{i \in I} (A_i \cup B) \qquad \text{and} \qquad \Bigl(\bigcup_{i \in I} A_i\Bigr) \cap B = \bigcup_{i \in I} (A_i \cap B)$$

De Morgan's laws:

$$\Bigl(\bigcap_{i \in I} A_i\Bigr)^c = \bigcup_{i \in I} A_i^c \qquad \text{and} \qquad \Bigl(\bigcup_{i \in I} A_i\Bigr)^c = \bigcap_{i \in I} A_i^c$$

Intersection of sets is also *associative*, that is, it does not matter how we bracket an expression. For instance $A \cap (B \cap C) = (A \cap B) \cap C$. Unions have a similar property.

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Questions to complete during the tutorial

1. Let $A, B, C, \dots$ be subsets of a set X. Simplify the following expressions.

(a) $(A \cup B) \cap A^c$

(b) $S \cap (A \cup B) \cap A$ if $A \cap B = \emptyset$.

(c) $(A^c \cup B^c) \cap (A^c \cup B)$

(d) $(A \cap B) \cup (A \cap B^c)$

2. Let $A_n \in X$ for $n \in \mathbb{N}$. Set $B_0 := A_0$ and $B_n := A_n \cap (A_0 \cup A_1 \cup \cdots \cup A_{n-1})^c$ for $n \geq 1$.

(a) Prove by induction by n that $\bigcup_{k=0}^n B_k = \bigcup_{k=0}^n A_k$ for all $n \geq 0$.

(b) Prove that $B_j \cap B_k = \emptyset$ if $j \neq k$.

3. Suppose that X, Y are sets and $f\colon X \to Y$. For $A \subset Y$ set $f^{-1}[A] := \{x \in X : f(x) \in A\}$.

(a) Prove that $\left(f^{-1}[A]\right)^c = f^{-1}[A^c]$.

(b) Suppose that $A_i \subseteq Y$ for all $i \in I$, where I is an arbitrary index set. Prove that

$$\bigcup_{i \in I} f^{-1}[A_i] = f^{-1}\Big[\bigcup_{i \in I} A_i\Big] \qquad \text{and} \qquad \bigcap_{i \in I} f^{-1}[A_i] = f^{-1}\Big[\bigcap_{i \in I} A_i\Big].$$

4. Suppose that X, Y are sets and $f\colon X \to Y$ a function.

(a) If $\mathcal{A}$ is a σ-algebra in Y, prove that $\mathcal{A}_f := \{f^{-1}[A] : A \in \mathcal{A}\}$ is a σ-algebra in X.

(b) If $\mathcal{A}$ is a σ-algebra in X, prove that $\mathcal{A}_1 := \{A \subseteq Y : f^{-1}[A] \in \mathcal{A}\}$ is a σ-algebra in Y.

Extra questions for further practice

5. Let $\mathcal{A}$ be the collection of all finite subsets of $\mathbb{R}$ and their complements. Show that $\mathcal{A}$ is an algebra, but not a σ-algebra.

6. Denote by m^* the Lebesgue outer measure in $\mathbb{R}^N$ as defined in lectures.

(a) Show that $\{x\}$ is measurable and $m^*(\{x\}) = 0$.

(b) Let $C \subseteq \mathbb{R}^N$ be a countable set. Show that $m^*(C) = 0$.

7. Let $A \subset \mathbb{R}^N$ and $\delta > 0$. Consider the following set functions:

$$m^*(A) = \inf\Big\{\sum_{k=0}^{\infty} \operatorname{vol}(R_k) \colon A \subseteq \bigcup_{k=0}^{\infty} R_k,\ R_k \text{ open rectangles}\Big\},$$

$$\bar{m}^*(A) = \inf\Big\{\sum_{k=0}^{\infty} \operatorname{vol}(R_k) \colon A \subseteq \bigcup_{k=0}^{\infty} R_k,\ R_k \text{ closed rectangles}\Big\},$$

$$m^*_\delta(A) = \inf\Big\{\sum_{k=0}^{\infty} \operatorname{vol}(R_k) \colon A \subseteq \bigcup_{k=0}^{\infty} R_k,\ R_k \text{ rectangles with } \operatorname{diam}(R_k) < \delta\Big\},$$

For a set $A \subseteq \mathbb{R}^N$ define the diameter $\operatorname{diam}(A) := \sup_{x,y \in A} \|y - x\|$.

(a) Prove that $m^*(A) = \bar{m}^*(A)$ for all $A \subseteq \mathbb{R}^N$.

(b) Let R be a rectangle and $\varepsilon, \delta > 0$. Show that R can be covered by countably many rectangles R_j with $\operatorname{diam}(R_j) < \delta$ such that $\operatorname{vol}(R) + \varepsilon \geq \sum_{j=0}^{\infty} \operatorname{vol}(R_j)$.

(c) Show that $m^*(A) = m_\delta(A)$ for all $A \subseteq \mathbb{R}^N$.

8. (a) Prove the distributive laws.

(b) Prove de Morgan's laws.

Challenge questions (optional)

9. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a right continuous increasing function, that is, $F(t) = \lim_{s\to t+} F(s)$ for all $t \in \mathbb{R}$. For $a \leq b$ define $\nu_F((a,b]) := F(b) - F(a)$. If $A \subseteq \mathbb{R}$ let

$$\mu_F^*(A) := \inf\Big\{\sum_{k=0}^{\infty} \nu_F(I_k)\colon I_k = (a_k, b_k], A \subseteq \bigcup_{k=0}^{\infty} I_k\Big\}.$$

(a) Show that μ_F^* is an outer measure (called the *Lebesgue-Stieltjes outer measure*)

(b) Show that $\mu_F^*((a,b]) \leq \nu_F((a,b]) = F(b) - F(a)$.

*(c) Show that $\mu_F^*((a,b]) \geq \nu_F((a,b]) = F(b)-F(a)$ and hence from (b) $\mu_F^*((a,b]) \geq \nu_F((a,b])$.

10. Let $\mu\colon \mathcal{A} \to [0,\infty]$ be a Borel measure measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. Define the function $F(t) := \mu((-\infty, t])$.

The function F is called the *distribution function* of the measure μ and is often used in probability theory.

(a) Show that F is increasing and right continuous.

(b) show that $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to\infty} F(t) = 1$.

(c) Show that $\mu_F(\{t\}) = F(t) - \lim_{s\to t-} F(t)$ (The height of the jump of F at t).

(d) Show that a bounded increasing function on $\mathbb{R}$ can have at most countably many discontinuities.