The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 1 (Week 2)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Outcomes

After completing this tutorial you should

(1) understand the basic rules of set theory as needed for measure theory.

(2) be able to work with inverse images of functions.

(3) use the rules of set theory and inverse images in relation to σ-algebras.

(4) be able to work with the definition of the Lebesgue outer measure.

For your reference we collect some facts on set theory. We always look at a base set X and subsets thereof. Here are some precise definition:

- $A \subseteq X$ is a subset of X if $x \in A$ implies $x \in X$. The collection of all subsets of X is called the *power set* of X. We denote the power set of X by $\mathcal{P}(X)$.
- If $A, B \subseteq X$ we write $A \subseteq B$ if $x \in B$ for every $x \in A$ and $A = B$ if $A \subseteq B$ and $B \subseteq A$.
- Let I be an arbitrary index set, for instance $I = \{1, 2, \dots, n\}$ finite, $I = \mathbb{N}$ countable or $I = \mathbb{R}$ uncountable. There is no restriction on the cardinality of I. Suppose that for every $i \in I$ there is a set $A_i \subseteq X$. We define the *union*

$$\bigcup_{i\in I} A_i := \{x \in X : x \in A_i \text{ for some } i \in I\}$$

and the *intersection*

$$\bigcap_{i\in I} A_i := \{x \in X : x \in A_i \text{ for all } i \in I\}.$$

- We define the *complement* A^c of $A \subseteq X$ by

$$A^c := X \setminus A := \{x \in X : x \notin A\}.$$

For $X \setminus A$ we also say "X minus A".

As in algebra, there are rules on how to do computations with sets. We state the most basic ones below. We assume that $A_i \subseteq X$ for all $i \in I$ and $B \subseteq X$, where I is an arbitrary index set.

Distributive laws:

$$\Bigl(\bigcap_{i\in I} A_i\Bigr) \cup B = \bigcap_{i\in I} (A_i \cup B) \qquad \text{and} \qquad \Bigl(\bigcup_{i\in I} A_i\Bigr) \cap B = \bigcup_{i\in I} (A_i \cap B)$$

De Morgan's laws:

$$\Bigl(\bigcap_{i\in I} A_i\Bigr)^c = \bigcup_{i\in I} A_i^c \qquad \text{and} \qquad \Bigl(\bigcup_{i\in I} A_i\Bigr)^c = \bigcap_{i\in I} A_i^c$$

Intersection of sets is also *associative*, that is, it does not matter how we bracket an expression. For instance $A \cap (B \cap C) = (A \cap B) \cap C$. Unions have a similar property.

Questions to complete during the tutorial

1. Let $A, B, C, \dots$ be subsets of a set X. Simplify the following expressions.

(a) $(A \cup B) \cap A^c$

Solution: Using the laws from above and the fact that $A \cap A^c = \emptyset$

$$(A \cup B) \cap A^c = (A \cap A^c) \cup (B \cap A^c) = B \cap A^c.$$

(b) $S \cap (A \cup B) \cap A$ if $A \cap B = \emptyset$.

Solution: $S \cap (A \cup B) \cap A = S \cap \big((A \cap A) \cup (B \cap A)\big) = S \cap A$

(c) $(A^c \cup B^c) \cap (A^c \cup B)$

Solution: $(A^c \cup B^c) \cap (A^c \cup B) = A^c \cap (B^c \cup B) = A^c \cap X = A^c$

(d) $(A \cap B) \cup (A \cap B^c)$

Solution: $(A \cap B) \cup (A \cap B^c) = A \cup (B \cap B^c) = A \cup \emptyset = A$

2. Let $A_n \in X$ for $n \in \mathbb{N}$. Set $B_0 := A_0$ and $B_n := A_n \cap (A_0 \cup A_1 \cup \dots \cup A_{n-1})^c$ for $n \geq 1$.

(a) Prove by induction by n that $\bigcup_{k=0}^n B_k = \bigcup_{k=0}^n A_k$ for all $n \geq 0$.

Solution: The case $n = 0$ is obvious since $A_0 = B_0$ by definition.
Assume that the identity holds for n. By definition of B_{n+1} and the distributive laws

$$\bigcup_{j=0}^{n+1} B_j = \Big(\bigcup_{j=0}^{n} B_j\Big) \cup B_{n+1} = \Big(\bigcup_{j=0}^{n} A_j\Big) \cup \Big(A_{n+1} \cap \Big(\bigcup_{j=0}^{n} A_j\Big)^c\Big)$$
$$= \Big(\Big(\bigcup_{j=0}^{n} A_j\Big) \cup A_{n+1}\Big) \cup X = \bigcup_{j=0}^{n+1} A_j$$

as claimed. Here we used that $X = \bigcup_{j=0}^n A_j \cup \bigcup_{j=0}^n A_j\Big)^c$.

(b) Prove that $B_j \cap B_k = \emptyset$ if $j \neq k$.

Solution: Without loss of generality we can assume that $k > j$ (otherwise we interchange the roles of j and k). By de Morgan's law and since $A_j \cap A_j^c = \emptyset$ we get

$$B_k \cap B_j = A_k \cap A_0^c \cap \dots \cap A_{k-1}k^c \cap A_j \cap A_0^c \cap \dots \cap A_j^c = (A_j \cap A_j^c) \cap A_k \cap A_0^c \cap \dots \cap A_{k-1} = \emptyset.$$

3. Suppose that X, Y are sets and $f\colon X \to Y$. For $A \subset Y$ set $f^{-1}[A] := \{x \in X \colon f(x) \in A\}$.

(a) Prove that $\big(f^{-1}[A]\big)^c = f^{-1}[A^c]$.

Solution: For every $x \in X$ either $f(x) \in A$ or $f(x) \in A^c$, but not both. Hence $X = f^{-1}[A] \cup f^{-1}[A^c]$ is a disjoint union and so our claim follows.

(b) Suppose that $A_i \subseteq Y$ for all $i \in I$, where I is an arbitrary index set. Prove that

$$\bigcup_{i \in I} f^{-1}[A_i] = f^{-1}\big[\bigcup_{i \in I} A_i\big] \qquad \text{and} \qquad \bigcap_{i \in I} f^{-1}[A_i] = f^{-1}\big[\bigcap_{i \in I} A_i\big].$$

Solution: We prove the first identity. Let $x \in f^{-1}\Big[\bigcup_{j=0}^\infty A_j\Big]$. Then $f(x) \in \bigcup_{j=0}^\infty A_j$, and therefore $f(x) \in A_j$ for some j. But this means that $x \in f^{-1}(A_j)$, and so $x \in \bigcup_{j=0}^\infty f^{-1}[A_j]$, showing that $f^{-1}\Big[\bigcup_{j=0}^\infty A_j\Big] \subseteq \bigcup_{j=0}^\infty f^{-1}[A_j]$. If on the other hand $x \in \bigcup_{j=0}^\infty f^{-1}[A_j]$, then $x \in f^{-1}[A_j]$ for some $j \in \mathbb{N}$, and so $f(x) \in A_j \subseteq \bigcup_{j=0}^\infty A_j$. Hence $x \in f^{-1}\Big[\bigcup_{j=0}^\infty A_j\Big]$ and so $\bigcup_{j=0}^\infty f^{-1}[A_j] \subseteq f^{-1}\Big[\bigcup_{j=0}^\infty A_j\Big]$ as required.
The other identity follows in a similar way, replacing "some $j \in I$" by "all $j \in I$" in the above proof.

4. Suppose that X, Y are sets and $f\colon X \to Y$ a function.

(a) If $\mathcal{A}$ is a σ-algebra in Y, prove that $\mathcal{A}_f := \{f^{-1}[A]\colon A \in \mathcal{A}\}$ is a σ-algebra in X.

Solution: (i) Clearly $\emptyset = f^{-1}[\emptyset] \in \mathcal{A}_f$.
(ii) By Question 3 we have $(f^{-1}[A])^c = f^{-1}[A^c]$. Since $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$, we get $(f^{-1}[A])^c \in \mathcal{A}_f$ for every $A \in \mathcal{A}$.
(iii) Let $B_j \in \mathcal{A}_f$, $j \in \mathbb{N}$. Then there exist $A_j \in \mathcal{A}$ with $B_j = f^{-1}[A_j]$. Hence from Question 3

$$\bigcup_{j=0}^{\infty} B_j = \bigcup_{j=0}^{\infty} f^{-1}[A_j] = f^{-1}\Big[\bigcup_{j=0}^{\infty} A_j\Big] \in \mathcal{A}_f.$$

(b) If $\mathcal{A}$ is a σ-algebra in X, prove that $\mathcal{A}_1 := \{A \subseteq Y\colon f^{-1}[A] \in \mathcal{A}\}$ is a σ-algebra in Y.

Solution: (i) Clearly $\emptyset = f^{-1}[\emptyset]$, so $\emptyset \in \mathcal{A}_1$.
(ii) Suppose that $A \subseteq Y$ is such that $f^{-1}[A] \in \mathcal{A}$. Since $f^{-1}[A] \in \mathcal{A}$ implies $f^{-1}[A^c] = \big(f^{-1}[A]\big)^c \in \mathcal{A}$ by Question 3, we get $A^c \in \mathcal{A}_1$.
(iii) Let $A_j \subseteq Y$ with $f^{-1}[A_j] \in \mathcal{A}$ for all $j \in \mathbb{N}$. Then there exist $A_j \in \mathcal{A}$ with $B_j = f^{-1}[A_j]$. Hence by Question 3

$$= f^{-1}\Big[\bigcup_{j=0}^{\infty} A_j\Big] = \bigcup_{j=0}^{\infty} f^{-1}[A_j] \in \mathcal{A},$$

so $\bigcup_{j=0}^{\infty} A_j \in \mathcal{A}_1$.

Extra questions for further practice

5. Let $\mathcal{A}$ be the collection of all finite subsets of $\mathbb{R}$ and their complements. Show that $\mathcal{A}$ is an algebra, but not a σ-algebra.

Solution: Clearly $\emptyset \in \mathcal{A}$. If A is finite, its complement is in $\mathcal{A}$ by definition, and vice versa. If $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$. Indeed, if A, B are finite, then $A \cup B$ is finite. If A is finite and B is the complement of a finite set, then $A \cup B$ is the complement of a finite set. Finally, if A and B are complements of finite sets, so is $A \cup B$.

We show that $\mathcal{A}$ is not a σ-algebra. Let $A_n := \{n\}$. Then $A_n \in \mathcal{A}$ for all $n \in \mathbb{Z}$. However, $\bigcup_{n\in\mathbb{Z}} A_n = \mathbb{Z}$ is not in $\mathcal{A}$. Hence $\mathcal{A}$ is not a σ-algebra.

6. Denote by m^* the Lebesgue outer measure in $\mathbb{R}^N$ as defined in lectures.

(a) Show that $\{x\}$ is measurable and $m^*(\{x\}) = 0$.

Solution: Fix $\varepsilon > 0$ and let

$$R_0 := (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon)$$

and $R_k = \emptyset$ for all $k \geq 1$. Then $\{x\} \subseteq \bigcup_{k=0}^{\infty} R_k$ and

$$m^*(\{x\}) \leq \sum_{k=0}^{\infty} \operatorname{vol}(R_k) = (2\varepsilon)^N$$

Since the above is true for all $\varepsilon > 0$ we get $m^*(\{x\}) = 0$. From lectures we know that sets of outer measure zero are measurable.

(b) Let $C \subseteq \mathbb{R}^N$ be a countable set. Show that $m^*(C) = 0$.

Solution: Since C is countable we can enumerate its elements by $x_0, x_1, x_2, \ldots$. By the countable subadditivity of outer measures and the previous part

$$0 \leq m^*(C) \leq \sum_{k=0}^{\infty} m^*(\{x_k\}) = 0.$$

7. Let $A \subset \mathbb{R}^N$ and $\delta > 0$. Consider the following set functions:

$$m^*(A) = \inf\Big\{\sum_{k=0}^{\infty} \operatorname{vol}(R_k) \colon A \subseteq \bigcup_{k=0}^{\infty} R_k,\ R_k \text{ open rectangles}\Big\},$$
$$\bar{m}^*(A) = \inf\Big\{\sum_{k=0}^{\infty} \operatorname{vol}(R_k) \colon A \subseteq \bigcup_{k=0}^{\infty} R_k,\ R_k \text{ closed rectangles}\Big\},$$
$$m_\delta^*(A) = \inf\Big\{\sum_{k=0}^{\infty} \operatorname{vol}(R_k) \colon A \subseteq \bigcup_{k=0}^{\infty} R_k,\ R_k \text{ rectangles with } \operatorname{diam}(R_k) < \delta\Big\},$$

For a set $A \subseteq \mathbb{R}^N$ define the diameter $\operatorname{diam}(A) := \sup_{x,y\in A} \|y - x\|$.

(a) Prove that $m^*(A) = \bar{m}^*(A)$ for all $A \subseteq \mathbb{R}^N$.

Solution: Let $A \subset \mathbb{R}^N$ and R_k, $k \in \mathbb{N}$, open rectangles with $A \subseteq \bigcup_{k=0}^{\infty} R_k$. Then also $A \subseteq \bigcup_{k=0}^{\infty} \bar{R}_k$, that is, every cover with open rectangles induces a cover with closed rectangles. Hence by definition $\bar{m}^*(A) \leq m^*(A)$. To prove the opposite inequality fix $\varepsilon > 0$ and choose open rectangles such that $A \subseteq \bigcup_{k=0}^{\infty} \bar{R}_k$ and

$$\sum_{k=0}^{\infty} \operatorname{vol}(\bar{R}_k) < \bar{m}^*(A) + \varepsilon.$$

By enlarging R_k slightly on each side we can construct open rectangles $R_{\varepsilon k}$ such that $\bar{R}_k \subseteq R_{\varepsilon k}$ and $\operatorname{vol}(R_{\varepsilon k}) < \operatorname{vol}(R_k) + \varepsilon/2^{k+1}$. Then clearly $A \subseteq \bigcup_{k=0}^{\infty} R_k$ and

$$m^*(A) \leq \sum_{k=0}^{\infty} \operatorname{vol}(R_{\varepsilon k}) < \sum_{k=0}^{\infty} \operatorname{vol}(R_k) + \sum_{k=0}^{\infty} \frac{\varepsilon}{2^{k+1}} = \sum_{k=0}^{\infty} \operatorname{vol}(R_k) + \varepsilon < \bar{m}^*(A) + \varepsilon.$$

Since the above works for every choice of $\varepsilon > 0$ we conclude that $m^*(A) \leq m^*(A)$. Hence equality follows.

(b) Let R be a rectangle and $\varepsilon, \delta > 0$. Show that R can be covered by countably many rectangles R_j with $\operatorname{diam}(R_j) < \delta$ such that $\operatorname{vol}(R) + \varepsilon \geq \sum_{j=0}^{\infty} \operatorname{vol}(R_j)$.

Solution: For $j = (j_1, j_2, \dots, j_N) \in \mathbb{R}^N$ and $k \in \mathbb{N}$ define the open cubes

$$Q_{j,k} := \Big(\frac{j_1}{2^k}, \frac{j_1 + 1}{2^k}\Big) \times \dots \times \Big(\frac{j_N}{2^k}, \frac{j_N + 1}{2^k}\Big)$$

They define a countable decomposition of $\mathbb{R}^N$ such that for every $k \in \mathbb{N}$, we have $\bigcup_{j\in\mathbb{Z}^N} \bar{Q}_{j,k} = \mathbb{R}^N$. We now choose k such that $\operatorname{diam}(Q_{j,k}) < \delta$ for all $j \in \mathbb{Z}^N$. Given a rectangle R we then define rectangles by $Q_{j,k} \cap R \neq \emptyset$. Denote that collection of rectangles by R'_i, $i \in I$. That collection is at most countable, $R \subseteq \bigcup_{i\in I} \bar{R}'_i$ and $R'_i \cap R'_j = \emptyset$ for $i \neq j$. Hence in particular

$$\operatorname{vol}(R) = \sum_{j=0}^{\infty} \operatorname{vol}(R'_j).$$

Given $\varepsilon > 0$ we choose open rectangles R_j such that $\bar{R}'_j \subseteq R_j$ and $\operatorname{vol}(R_j) \leq \operatorname{vol}(R'_j) + \varepsilon/2^{j+1}$. Then

$$\operatorname{vol}(R) + \varepsilon = \sum_{j\in I} \operatorname{vol}(R'_j) + \varepsilon \geq \sum_{j=\in I} \Big(\operatorname{vol}(R'_j) + \frac{\varepsilon}{2^{j+1}}\Big) \geq \sum_{j=\in I} \operatorname{vol}(R_j)$$

as required.

(c) Show that $m^*(A) = m_\delta(A)$ for all $A \subseteq \mathbb{R}^N$.

Solution: Clearly $m^*(A) \leq m_\delta(A)$ from the definition and properties of an infimum. To show the other inequality fix $\varepsilon > 0$ and choose rectangles R_j, $j \in \mathbb{N}$ such that $A \subset \bigcup_{j=1}^\infty R_j$ and

$$m^*(A) + \varepsilon > \sum_{k=0}^{\infty} \mathrm{vol}(R_k)$$

By part (b), for each rectangle R_k we can choose a covering R_{jk} of rectangles with $\mathrm{diam}(R_{jk}) < \delta$ and $\sum_{j=0}^\infty \mathrm{vol}(R_{jk}) < \mathrm{vol}(R_k) + \varepsilon/2^{k+1}$. Then only choose those of R_{j_k} which have non-empty intersection with A. This collection is countable and we denote it by R_i', $i \in \mathbb{N}$. Then clearly

$$A \subset \bigcup_{i=0}^{\infty} R_i' \subset A_\delta$$

and

$$\sum_{i=0}^{\infty} \mathrm{vol}(R_i') \leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \mathrm{vol}(R_{jk}) < \sum_{i=0}^{\infty}\Bigl(\mathrm{vol}(R_k) + \frac{\varepsilon}{2^{k+1}}\Bigr) = \varepsilon + \sum_{i=0}^{\infty} \mathrm{vol}(R_k).$$

Putting everything together we get

$$m_\delta^*(A) \leq \sum_{i=0}^{\infty} \mathrm{vol}(R_i') < \varepsilon + \sum_{i=0}^{\infty} \mathrm{vol}(R_k) < m^*(A) + 2\varepsilon.$$

Since the above argument works for every choice of $\varepsilon > 0$ we get $m_\delta^*(A) \leq m^*(A)$, hence proving equality.

8. (a) Prove the distributive laws.

Solution: For the proof we extensively use the definition of the intersection and the union of sets without saying so every time.

(i) Assume that $x \in \Bigl(\bigcap_{i\in I} A_i\Bigr) \cup B$. Then $x \in \bigcap_{i\in I} A_i$ or $x \in B$. In the first case $x \in A_i$ for every $i \in I$, and so $x \in A_i \cup B$ for all $i \in I$. In the second case $x \in B$, so $x \in A_i \cup B$ for all $i \in I$ again. Hence in both cases $x \in \bigcap_{i\in I}(A_i \cup B)$. This shows that

$$\Bigl(\bigcap_{i\in I} A_i\Bigr) \cup B \subseteq \bigcap_{i\in I}(A_i \cup B).$$

To prove the opposite inclusion assume that $x \in \bigcap_{i\in I}(A_i \cup B)$. Then $x \in A_i \cap B$ for all $i \in I$. If $x \notin B$, then $x \in A_i$ for all $i \in I$, so $x \in \bigcap_{i\in I} A_i$. In the other case $x \in B$. In either case $x \in \Bigl(\bigcap_{i\in I} A_i\Bigr) \cup B$ by definition of a union of sets. Hence

$$\Bigl(\bigcap_{i\in I} A_i\Bigr) \cup B \supseteq \bigcap_{i\in I}(A_i \cup B).$$

Since we have proved both inclusions, equality follows.

(ii) Suppose that $x \in \Bigl(\bigcup_{i\in I} A_i\Bigr) \cap B$. Then $x \in \bigcup_{i\in I} A_i$ and $x \in B$. Hence there exists $j \in I$ with $x \in A_j$. Therefore $x \in A_j \cap B$ and so $x \in \Bigl(\bigcup_{i\in I} A_i\Bigr) \cap B$. Hence

$$\Bigl(\bigcup_{i\in I} A_i\Bigr) \cap B \subseteq \bigcap_{i\in I}(A_i \cap B).$$

to prove the opposite inclusion assume that $x \in \bigcap_{i\in I}(A_i \cap B)$. Then there exists $j \in I$ such that $x \in A_j \cap B$ and therefore $x \in B$ and $x \in A_j$. Hence $x \in \bigcup_{i\in I} A_i$ and so $x \in \Bigl(\bigcup_{i\in I} A_i\Bigr) \cap B$. This proves the opposite inclusion

$$\Bigl(\bigcup_{i\in I} A_i\Bigr) \cap B \supseteq \bigcap_{i\in I}(A_i \cap B).$$

(b) Prove de Morgan's laws.

Solution:

(i) Suppose that $x \in \left(\bigcap_{i\in I} A_i\right)^c$. Then $x \notin \bigcap A_i$, so there exists $j \in I$ such that $x \notin A_j$, that is, $x \in A_j^c$. Hence $x \in \bigcup_{i\in I} A_j^c$ and so

$$\Big(\bigcap_{i\in I} A_i\Big)^c \subseteq \bigcup_{i\in I} A_i^c.$$

To prove the opposite inclusion let $x \in \bigcup_{i\in I} A_i^c$. Then there exists $j \in I$ such that $x \in A_j^c$, that is, $x \notin A_j$. Therefore $x \notin \bigcap_{i\in I} A_i$ and so $x \in \left(\bigcap_{i\in I} A_i\right)^c$. Hence

$$\Big(\bigcap_{i\in I} A_i\Big)^c \supseteq \bigcup_{i\in I} A_i^c$$

and equality follows.

(ii) We prove the second law by using the first one by using that $(B^c)^c = B$ for every set:

$$\bigcap_{i\in I} A_i^c = \Big(\Big(\bigcup_{i\in I} A_i^c\Big)^c\Big)^c = \Big(\bigcup_{i\in I} (A_i^c)^c\Big)^c = \Big(\bigcup_{i\in I} A_i\Big)^c$$

Challenge questions (optional)

9. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a right continuous increasing function, that is, $F(t) = \lim_{s\to t+} F(s)$ for all $t \in \mathbb{R}$. For $a \leq b$ define $\nu_F((a,b]) := F(b) - F(a)$. If $A \subseteq \mathbb{R}$ let

$$\mu_F^*(A) := \inf\Big\{\sum_{k=0}^{\infty} \nu_F(I_k)\colon I_k = (a_k, b_k], A \subseteq \bigcup_{k=0}^{\infty} I_k\Big\}.$$

(a) Show that μ_F^* is an outer measure (called the *Lebesgue-Stieltjes outer measure*)

Solution: We first prove that $\mu_F^*(\emptyset) = 0$. For that we choose $I_k := (0,0] = \emptyset$ for all $k \in \mathbb{N}$. Since $\emptyset \subseteq \bigcup_{k\in\mathbb{N}}(0,0] = \emptyset$ the definition of μ_F^* implies

$$0 \leq \mu_F^*(\emptyset) \ \leq \sum_{k=0}^{\infty} \nu_F((0,0]) \leq \sum_{k=0}^{\infty}\big(F(0) - F(0)\big) = 0.$$

Hence $\mu_F^*(\emptyset) = 0$. We next show the countable sub-additivity. Let $A, A_k \subseteq \mathbb{R}$, $k \in \mathbb{N}$ and $A \subseteq \bigcup_{k=0}^{\infty} A_k$. Let $\varepsilon > 0$ and choose intervals I_{jk} such that $A_j \subseteq \bigcup_{j=1}^{\infty} I_{jk}$ and $\mu_F^*(A_k) + \varepsilon/2^{k+1} > \sum_{j=0}^{\infty} \nu_F(I_{jk})$. Then

$$A \subseteq \bigcup_{k=0}^{\infty}\bigcup_{j=1}^{\infty} I_{jk}$$

and hence

$$\mu_F^*(A) \leq \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \nu_F(I_{jk}) \leq \sum_{k=0}^{\infty}\Big(\mu_F^*(A_k) + \frac{\varepsilon}{2^{k+1}}\Big) = \sum_{k=0}^{\infty} \mu_F^*(A_k) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we get $\mu_F^*(A) \leq \sum_{k=0}^{\infty} \mu_F^*(A_k)$. This completes the proof.

(b) Show that $\mu_F^*((a,b]) \leq \nu_F((a,b]) = F(b) - F(a)$.

Solution: Fix an interval $(a,b]$. We can produce a cover by $I_0 := (a,b]$ and the intervals $I_k = (0,0] = \emptyset$ for $k \geq 1$. Then $(a,b] \subseteq \bigcup_{k=0}^{\infty} I_k$ and

$$\mu_F^*((a,b]) \leq \sum_{k=0}^{\infty} \nu_F(I_k) \leq F(b) - F(a) + \sum_{k=0}^{\infty}\big(F(0) - F(0)\big) = F(b) - F(a) = \nu_F((a,b])$$

as claimed.

*(c) Show that $\mu_F^*((a,b]) \geq \nu_F((a,b]) = F(b) - F(a)$ and hence from (b) $\mu_F^*((a,b]) \geq \nu_F((a,b])$.

Solution: The opposite inequality is a bit more tricky. We have to show that for *every possible cover* $(a,b] \subseteq \bigcup_{k=0}^{\infty} I_k$ with intervals of the form $I_k = (a_k, b_k]$ we have

$$F(b) - F(a) \leq \sum_{k=0}^{\infty} \nu_F(I_k). \tag{1}$$

We distinguish two cases.

(i) First assume that finitely many of the I_k's cover $(a,b]$. Renumbering we can assume that $(a,b] \subseteq \bigcup_{k=0}^{n} I_k$. Now order all endpoints of these k intervals in $(a,b]$ and denote them by $a = \xi_0 < \xi_1 < \cdots < \xi_m = b$. Then for each $i = 1, \dots, m$ the interval $J_i := (\xi_{i-1}, \xi_i] \subseteq I_k$ for some k. Hence $\nu_F(J_i) \leq \nu_F(I_k)$ and so

$$\sum_{k=0}^{\infty} \nu_F(I_k) \geq \sum_{k=0}^{n} \nu_F(I_k) \geq \sum_{i=1}^{m} \nu_F(J_i) = \sum_{i=1}^{m} (F(\xi_i) - F(\xi_{i-1})) = F(b) - F(a).$$

(ii) If there is no finite sub-cover, the proof is more difficult. Fix $\varepsilon > 0$ and choose $\delta_k > 0$ such that $F(b_k + \delta_k) - F(b_k) < \varepsilon/2^{k+1}$. Set $I_{\varepsilon k} := (a_k, b_k + \delta_k]$. Then we have $(a,b] \subseteq \bigcup_{k=0}^{\infty} I_{\varepsilon k}$ and

$$\begin{aligned}\sum_{k=0}^{\infty} \nu_F(I_{\varepsilon k}) &= \sum_{k=0}^{\infty} \big(F(b_k + \delta_k) - F(a_k)\big) \\ &= \sum_{k=0}^{\infty} \big(F(b_k + \delta_k) - F(b_k)\big) + \sum_{k=0}^{\infty} \big(F(b_k) - F(a_k)\big) \\ &< \sum_{k=0}^{\infty} \nu_F(I_k) + \sum_{k=0}^{\infty} \frac{\varepsilon}{2^{k+1}} < \sum_{k=0}^{\infty} \nu_F(I_k) + \varepsilon. \end{aligned} \tag{2}$$

By the right continuity of F we can also choose $\delta > 0$ such that $F(a+\delta) - F(a) < \varepsilon$. Now clearly by construction, the open intervals $(a_k, b_k + \varepsilon)$ cover $[a + \delta, b]$. Since $[a + \delta, b]$ is compact the Heine-Borel theorem implies that there is a finite sub-cover. Renumbering we can assume these intervals are $I_{\varepsilon k}$, $k = 0, \dots, n$. Then $(a + \delta, b] \subseteq \bigcup_{k=0}^{n} I_{\varepsilon k}$. The argument used in (i) above implies

$$F(b) - F(a + \delta) \leq \sum_{k=0}^{n} \nu_F(I_{\varepsilon k}) \leq \sum_{k=0}^{\infty} \nu_F(I_{\varepsilon k}).$$

Using (2) and $F(a + \delta) - F(a) < \varepsilon$ we get

$$\begin{aligned} F(b) - F(a) &\leq \big(F(b) - F(a + \delta)\big) + \big(F(a + \delta) - F(a)\big) \\ &< \varepsilon + \sum_{k=0}^{\infty} \nu_F(I_{\varepsilon k}) < 2\varepsilon + \sum_{k=0}^{\infty} \nu_F(I_k). \end{aligned}$$

Since the above procedure works for every choice of $\varepsilon > 0$ we finally get (1).

10. Let $\mu\colon \mathcal{A} \to [0, \infty]$ be a Borel measure measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. Define the function $F(t) := \mu((-\infty, t])$.

The function F is called the *distribution function* of the measure μ and is often used in probability theory.

(a) Show that F is increasing and right continuous.

Solution: If $s \leq t$, then $(-\infty, s] \subseteq (-\infty, t]$ and so by the monotonicity of measures

$$F(s) = \mu\big((-\infty, s]\big) \leq \mu\big((-\infty, t]\big) = F(t),$$

so F is increasing. For the right continuity fix $t \in \mathbb{R}$ and let t_n be a decreasing sequence with $t_n \to t$. Then $(-\infty, t_n] \supseteq (-\infty, t_{n+1}]$ for all $n \in \mathbb{N}$ and

$$(-\infty, t] = \bigcap_{n \in \mathbb{N}} (-\infty, t_n].$$

Since $\mu((-\infty, t_0]) \leq \mu(\mathbb{R}) = 1 < \infty$ the monotonicity properties of measures implies

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \mu\big((-\infty, t_n]\big) = \mu\Big(\bigcap_{n\in\mathbb{N}} (-\infty, t_n]\Big) = \mu\big((-\infty, t]\big) = F(t).$$

Since this is true for all decreasing sequences with $t_n \to t$ it follows that F is right continuous.

(b) show that $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to\infty} F(t) = 1$.

Solution: Again by the monotonicity of measures

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \mu\big((-\infty, t_n]\big) = \mu\Big(\bigcap_{n\in\mathbb{N}} (-\infty, t_n]\Big) = \mu(\emptyset) = 0.$$

for every decreasing sequence $t_n \to -\infty$ and

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \mu\big((-\infty, t_n]\big) = \mu\Big(\bigcup_{n\in\mathbb{N}} (-\infty, t_n]\Big) = \mu(\mathbb{R}) = 1$$

for every increasing sequence $t_n \to \infty$. Hence $F(t) \to 0$ if $t \to -\infty$ and $F(t) \to 1$ if $t \to \infty$.

(c) Show that $\mu_F(\{t\}) = F(t) - \lim_{s\to t-} F(t)$ (The height of the jump of F at t).

Solution: Since the function is bounded and monotone $\lim_{s\to t-} F(s)$ exists at every point $t \in \mathbb{R}$. Now for every increasing sequence t_n with $t_n \to t$ we have from the monotonicity property

$$\mu\{t\} = \mu\Big(\bigcap_{n\in\mathbb{N}} (t_n, t]\Big) = \lim_{n\to\infty} \mu\big((t_n, t]\big) = \lim_{n\to\infty} F(t_n) - F(t) = \lim_{s\to t-} F(s) - F(t)$$

as claimed.

(d) Show that a bounded increasing function on $\mathbb{R}$ can have at most countably many discontinuities.

Solution: Since F is monotone, left and right limits exist at every point. Let $S_n := \{t \in \mathbb{R} \colon \lim_{s\to 0+} F(s) - \lim_{s\to 0-} F(s) > 1/n\}$ be the set of discontinuities with a jump larger than $1/n$. This is a finite set for every n as otherwise the function is unbounded. The set of all discontinuities $\bigcup_{n\in\mathbb{N}} S_n$ is therefore at most countable.