Material covered

(1) Properties and constructions of outer measures
(2) Construction of measures
(3) Properties of the Lebesgue measure.

Outcomes

After completing this tutorial you should

(1) be able to construct outer measures and measures
(2) know the main invariance properties of the Lebesgue measure
(3) be able to work with dyadic decomposition to prove properties of the Lebesgue measure.

Summary of essential material

The construction of measures from outer measures

Let $X$ be an arbitrary set. An outer measure on $X$ is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ with the following properties:

(i) $\mu^*(\emptyset) = 0$

(ii) If $A, A_k \subseteq X$ with $A \subseteq \bigcup_{k=0}^{\infty} A_k$, then $\mu^*(A) \leq \sum_{k=0}^{\infty} \mu^*(A_k)$ (countable sub-additivity).

The sets in $\mathcal{A}$ are called measurable sets with respect to $\mu$ (or $\mu^*$). Note that an outer measure is always defined on the power set $\mathcal{P}(X)$, that is, for all subsets of $X$. Even if the sets $A_k$ are disjoint, we only require countable sub-additivity, not additivity. Carathéodory’s theorem states that

$$\mathcal{A} := \{ A \subseteq X : \mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \text{ for all } S \subseteq X \}$$

is a $\sigma$-algebra, and that $\mu := \mu^*|_{\mathcal{A}} : \mathcal{A} \to [0, \infty]$ is a measure. We call $\mu$ the measure induced by the outer measure $\mu^*$.

The Lebesgue measure

The most important application of Carathéodory’s theorem for us is the construction of the Lebesgue measure. The Lebesgue measure in $\mathbb{R}^N$, that is, representing $N$-dimensional volume in $\mathbb{R}^N$, is the measure induced by the Lebesgue outer measure given by

$$m_N^*(A) := \inf \left\{ \sum_{k=0}^{\infty} \text{vol}_N(R_k) : R_k \text{ open rectangles, } A \subseteq \bigcup_{k=0}^{\infty} R_k \right\}.$$ 

The corresponding $\sigma$-algebra of $m_N^*$-measurable sets is called the Lebesgue $\sigma$-algebra $\mathcal{M}_N$. 

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Properties of the Lebesgue measure:

(i) The Lebesgue measure is a Borel measure. In particular, all open and closed sets are measurable, and all countable unions and intersections of open and closed sets are measurable.

(ii) The Lebesgue measure is the only translation invariant Borel measure on $\mathbb{R}^N$ such that all bounded sets have finite measure.

(iii) The Lebesgue measure is regular, that is:

$$m^*_N(A) = \inf \{ m_N(U) : A \subseteq U, U \text{ open} \}$$

for every $A \subseteq \mathbb{R}^N$.

$$m_N(A) = \sup \{ m_N(K) : A \supseteq K, K \text{ compact} \}$$

for every measurable $A \subseteq \mathbb{R}^N$.

**Dyadic decompositions** Every non-empty open set in $\mathbb{R}^N$ can be written as a disjoint union of countably many dyadic cubes. A dyadic cube is a cube of the form

$$Q_{n,k} = \left\{ \frac{k}{2^n} + \left(0, \frac{1}{2^n}\right)^N : k \in \mathbb{Z}^N \right\}.$$

Together with the regularity of the Lebesgue measure, dyadic decompositions are useful to prove many properties of the Lebesgue measure including uniqueness and invariance under rigid motion.

**Questions to complete during the tutorial**

1. Consider an arbitrary subset $A \subseteq \mathbb{R}^N$. In lectures it was shown that

$$m^*_N(A) = \inf \{ m_N(U) : A \subseteq U, U \text{ open} \}.$$  \hfill (1)

For simplicity, we throughout assume that $m^*_N(A) < \infty$.

(a) Show that for every $A \subseteq \mathbb{R}^N$ there exist open sets $U_n$ so that

$$m^*_N(A) = m_N\left(\bigcap_{n \in \mathbb{N}} U_n\right).$$

In particular, the Borel set $B = \bigcap_{n \in \mathbb{N}} U_n$ is such that $A \subseteq B$ and $m^*_N(A) = m_N(B)$.

(b) Show that for every Lebesgue measurable set $A \subseteq \mathbb{R}^N$ there exist open sets $U_n$ and a set $S$ with $m_N(S) = 0$ so that

$$A \cup S = \bigcap_{n \in \mathbb{N}} U_n.$$  \hfill (2)

(c) Let $A$ be a non-measurable set. Let $B$ be as in part (a) and define $S := B \cap A^c$. Show that $S$ is not measurable and that $m^*_N(S) > 0$.

2. Let $\Omega$ be the sample space obtained by doing an infinite sequence of coin tosses. Each such sequence can be represented in the form $(a_1, a_2, \ldots)$, where $a_k = 0$ or 1. A zero means head and a one means tail for instance. Such sequences can be interpreted as a binary expansions of the number $\alpha = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \in [0, 1]$.

There is one difficulty though: The association of a sequence $(a_1, a_2, \ldots)$ with the number $\alpha$ is not one-to-one. As $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ every sequence ending with zeros gives the same $\alpha$ as a sequence ending with ones. For instance,

$$(1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, \ldots)$$

$$(1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, \ldots)$$
represent the same \( a \). For this reason we discard all sequences that only contain finitely many ones. What we discard is a set of probability zero since the probability of any single sequence occurring is zero. There are only countably many sequences with finitely many ones since this set can be written as the union of countably many finite sequences. In particular we discard the zero sequence \((0, 0, 0, \ldots)\). With that identification, the sample space becomes \( \Omega = (0, 1] \).

(a) Set \( A_k := \{ (a_1, \ldots, a_k, \ldots) \in \Omega : a_k = 1 \} \). Sketch \( A_1, A_2, A_3 \) as subsets of \((0, 1]\) and then describe the sets \( A_k \) for general \( k \). What is the probability of \( A_k \), and how does it compare to the Lebesgue measure of \( A_k \)?

(b) Show that every dyadic interval \( I_{n,j} = (j/2^n, (j+1)/2^n], j = 0, \ldots, 2^{n-1} \), can be written as a finite intersection of the sets \( A_k \) and \( A_k^c \).

(c) Argue why the probability measure in the above situation coincides with the Lebesgue measure on \((0, 1]\).

3. The purpose of this question is to give an example of a set that is not Lebesgue measurable.

We define an equivalence relation on \( \mathbb{R}^N \) by \( x \sim y \) if \( x - y \in \mathbb{Q}^N \). The equivalence class of \( x \) is then given by \([x] := x + \mathbb{Q}^N\). Let now \( B \subseteq \mathbb{R}^N \) be the unit ball. Construct a set \( C \) as follows: For every equivalence class \([x]\) with \([x] \cap B \neq \emptyset\), choose precisely one \( y \in B \cap [x] \) to be in \( C \). Finally let \( I := 2B \cap \mathbb{Q}^N \).

(a) Show that the union \( \bigcup_{q \in I} (q + C) \) is disjoint.

(b) Show that \( B \subseteq \bigcup_{q \in I} (q + C) \subseteq 3B \).

(c) Show that \( m_N^*(C) > 0 \) and deduce that \( C \) is not measurable.

4. Let \((X, A, \mu)\) be a measure space. For every \( A \subseteq X \) we define

\[
\mu^*(A) := \inf \left\{ \sum_{k=1}^{\infty} \mu(A_k) : A_k \in A, k \in \mathbb{N}, A \subseteq \bigcup_{k=0}^{\infty} A_k \right\}
\]

(a) Show that \( \mu^* : P(X) \rightarrow [0, \infty] \) is an outer measure. To do so proceed similarly as for the Lebesgue outer measure.

(b) Prove that \( \mu^*(A) \leq \mu(A) \) for all \( A \in A \).

(c) Prove that \( \mu^*(A) = \mu(A) \) for all \( A \in \mathcal{A} \).

**Extra questions for further practice**

5. Consider a Lebesgue measurable set \( A \subseteq \mathbb{R}^N \).

(a) Show that \( m(A) = \sup \{ m(K) : K \subseteq A, K \text{ compact} \} \).

Hint: First assume that \( A \) is bounded and approximate \( S \cap A^c \) by open sets from outside for some compact set \( S \supseteq A \). Then look at unbounded \( A \).

(b) If \( A \) is Lebesgue measurable, show that there exists a Borel set \( C \) with \( C \subseteq A \) and \( m(A) = m(C) \).

6. Show that the Lebesgue outer measure of an arbitrary set \( A \subseteq \mathbb{R}^N \) is given by

\[
m_N^*(A) = \inf \left\{ \sum_{k=0}^{\infty} m_N(Q_k) : Q_k \text{ disjoint dyadic cubes and } A \subseteq \bigcup_{k \in \mathbb{N}} Q_k \right\}
\]

For the proof use the outer regularity of the Lebesgue measure and the dyadic decomposition of open sets.