The University of Sydney
School of Mathematics and Statistics

Tutorial 2 (Week 3)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
Lecturer: Daniel Daners

Material covered

(1) Properties of the Lebesgue integral.

(2) Elementary properties of measurable functions.

Outcomes

After completing this tutorial you should

(1) know the main invariance properties of the Lebesgue measure

(2) be able to work with dyadic decomposition to prove properties of the Lebesgue measure.

(3) be able to use the definition of measurable functions.

Questions to complete during the tutorial

1. Consider an arbitrary subset $A \subseteq \mathbb{R}^N$.

(a) Show that $m^*(A) = \inf\{m(U) \colon A \subseteq U, U \text{ open}\}$.
Hint: Use the definition of the Lebesgue outer measure and the fact that any union of open sets is open.

(b) Hence show that there exists a Borel measurable set B with $A \subseteq B$ and $m^*(A) = m(B)$.

2. From linear algebra it is known that every invertible matrix T can be written as a product $T = E_1E_2E_3 \dots E_m$, where E_k are elementary matrices corresponding to one of the following row operations:

(I) Multiply one row by $\lambda \in \mathbb{R}$;

(II) Add one row to another row.

Denote by Q the unit cube $(0,1) \times (0,1) \times \dots \times (0,1)$. For the proofs below you may use that the Lebesgue outer measure is translation invariant.

(a) If E is of the form (I), prove that $m_N(E(Q))) = |\lambda|$.

(b) If E is of the form (II), prove that $m_N(E(Q)) = 1$.

(c) If T is a linear transformation on $\mathbb{R}^N$, prove that $m_N(T(Q)) = |\det T|$.

(d) Let $U \subseteq \mathbb{R}^N$ be open and T a linear operator on $\mathbb{R}^N$. Show that $m_N(T(U)) = |\det T|$. Then argue why the same is true for any measurable set.
Hint: Use dyadic decomposition of open sets and Question 1.

(e) Why is $m_N(T(U)) = 0$ if T is not invertible?

3. Let $\mu \colon \mathcal{A} \to [0,\infty]$ be a measure on X. Suppose that $f = (f_1, \dots, f_N) \colon X \to \mathbb{R}^N$.

(a) Suppose that $A_1, \dots, A_N \subseteq \mathbb{R}$. Show that $f^{-1}[A_1 \times \dots \times A_N] = \bigcap_{k=1}^N f^{-1}[A_k]$.

(b) If f is measurable, show that f_k is measurable for every $k = 1, \dots, N$.

(c) If f_k are measurable for every $k = 1, \dots, N$, show that f is measurable.

Extra questions for further practice

4. Consider a Lebesgue measurable set $A \subseteq \mathbb{R}^N$.

(a) Show that $m(A) = \sup\{m(K) \colon K \subseteq A,\ K \text{ compact}\}$.

Hint: First assume that A is bounded and approximate $S \cap A^c$ by open sets from outside for some compact set $S \supseteq A$. Then look at unbounded A.

(b) If A is Lebesgue measurable, show that there exists a Borel set C with $C \subseteq A$ and $m(A) = m(C)$.

5. Let $\mu \colon \mathcal{A} \to [0, \infty]$ be a measure on the space X with $\mu(X) < \infty$. Suppose that $f \colon X \to \mathbb{R}$ is a μ-measurable function.

(a) Let $A_n := \{x \in X \colon |f(x)| > n\}$. Show that A_n is measurable and that $\mu(A_n) \to 0$ as $n \to \infty$.

(b) Given $\varepsilon > 0$, show that there exist a simple function and a set $U \subseteq X$ such that $\varphi \colon X \to \mathbb{R}$ with $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in U$ and $\mu(X \setminus U) < \varepsilon$.

Hint: Look at non-negative functions first.

Challenge questions (optional)

6. Suppose $S \subseteq [0, 1]$ is a set which is not Lebesgue measurable.

(a) Show that the indicator function 1_S is not measurable.

(b) Construct a function $f \colon [0, 1] \to \mathbb{R}$ which is not measurable, but the sets $\{x \in [0, 1] \colon f(x) = a\} = f^{-1}(\{a\})$ are measurable for all $a \in \mathbb{R}$.