THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 2 (Week 3)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) Properties of the Lebesgue integral.

(2) Elementary properties of measurable functions.

Outcomes

After completing this tutorial you should

(1) know the main invariance properties of the Lebesgue measure

(2) be able to work with dyadic decomposition to prove properties of the Lebesgue measure.

(3) be able to use the definition of measurable functions.

Questions to complete during the tutorial

1. Consider an arbitrary subset $A \subseteq \mathbb{R}^N$.

(a) Show that $m^*(A) = \inf\{m(U) : A \subseteq U, U \text{ open}\}$.

Hint: Use the definition of the Lebesgue outer measure and the fact that any union of open sets is open.

Solution: If $m^*(A) = \infty$ the claim is obvious. Hence assume that $m^*(A) < \infty$. By the monotonicity of outer measure we have $m^*(A) \leq m^*(U) = m(U)$ for all open sets U with $A \subseteq U$. Hence we need to show that for every $\varepsilon > 0$ there exists an open set U with $A \subseteq U$ such that $m^*(A) + \varepsilon \geq m(U)$. By definition of the Lebesgue outer measure there exist open rectangles R_k with $A \subseteq \bigcup_{k=0}^{\infty} R_k$ and

$$m^*(A) + \varepsilon > \sum_{k=0}^{\infty} \operatorname{vol}(R_k) = \sum_{k=0}^{\infty} m(R_k).$$

We set $U := \bigcup_{k=0}^{\infty} R_k$. Since arbitrary unions of open sets are open U is open. By the sub-additivity of measures we also have $m(U) \leq \sum_{k=0}^{\infty} m(R_k)$ and therefore

$$m^*(A) + \varepsilon > \sum_{k=0}^{\infty} m(R_k) \geq m(U),$$

so U is an open set as required.

(b) Hence show that there exists a Borel measurable set B with $A \subseteq B$ and $m^*(A) = m(B)$.

Solution: If $m^*(A) = \infty$, then the statement is obvious, so assume that $m^*(A) < \infty$. According to (a), for every $n \in \mathbb{N}$ there exists an open set $V_n \supset B$ such that $m^*(A) \leq m(V_n) \leq m^*(A) + 1/n$. Define $U_1 := V_1$, $U_{n+1} := V_{n+1} \cap U_n$. Then $m(U_n) \leq m(V_n)$ and $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Also

$$m^*(A) \leq m(U_n) \leq m(V_n) \leq m^*(A) + 1/n$$

for all $n \in \mathbb{N}$. We set $B := \bigcap_{n=1}^{\infty} U_n$. Then $A \subseteq U_n$ and by the monotonicity properties of measures and since $m(U_1) < \infty$

$$m^*(A) \leq \lim_{n\to\infty} m(U_n) = m\Big(\bigcap_{n=1}^{\infty} U_n\Big) = m(B) \leq m^*(A).$$

Since countable intersections of Borel sets are Borel sets, B is a Borel set as required.

2. From linear algebra it is known that every invertible matrix T can be written as a product $T = E_1E_2E_3 \dots E_m$, where E_k are elementary matrices corresponding to one of the following row operations:

(I) Multiply one row by $\lambda \in \mathbb{R}$;

(II) Add one row to another row.

Denote by Q the unit cube $(0,1) \times (0,1) \times \cdots \times (0,1)$. For the proofs below you may use that the Lebesgue outer measure is translation invariant.

(a) If E is of the form (I), prove that $m_N(E(Q))) = |\lambda|$.

Solution: Let

$$E = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \lambda & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

which multiplies the j-th component of a vector by λ. Hence if we have a rectangle $R = (a_1, b_1) \times \cdots \times (a_j, b_j) \times \cdots \times (a_N, b_N)$, then

$$E(R) = (a_1, b_1) \times \cdots \times (\lambda a_j, \lambda b_j) \times \cdots \times (a_N, b_N)$$

if $\lambda > 0$. If $\lambda < 0$ replace $(\lambda a_j, \lambda b_j)$ by $(\lambda b_j, \lambda a_j)$. Hence the volume of that the image is

$$\operatorname{vol}(E(R)) = |\lambda| \operatorname{vol}(R)$$

Hence for an arbitrary $A \subseteq \mathbb{R}^N$

$$\begin{aligned} m^*(E(A)) &= \inf\Big\{\sum_{k=0}^{\infty} \operatorname{vol}(R_k) \colon E(A) \subseteq \bigcup_{k=0}^{\infty} R_k\Big\} \\ &= \inf\Big\{|\lambda| \sum_{k=0}^{\infty} \operatorname{vol}(\tilde{R}_k) \colon E(A) \subseteq \bigcup_{k=0}^{\infty} \tilde{R}_k\Big\} = |\lambda| m^*(A). \end{aligned}$$

Here we use that there is a one to one correspondence between the rectancles R_k and $\tilde{R}_k$ by $E(\tilde{R}_k) = R_k$.

(b) If E is of the form (II), prove that $m_N(E(Q)) = 1$.

Solution: Add row i to row j. The corresponding linear transformation E acts like

$$E(x_1, \dots, x_N) = (x_1, \dots, x_{j-1}, x_j + x_i, x_{j+1}, \dots, x_N).$$

Geometrically E transforms Q by shearing it parallel to the j-th coordinate axis as shown in Figure 1 which shows a cross section of Q and $E(Q)$ in the plane spanned by the i-th and the j-th coordinate axis. The cross-section of Q is the square $ABCD$ and that of $E(Q)$ is the parallelogram $ACED$. The plane through line AC perpendicular to the ij-plane subdivides Q and $E(Q)$ into triangular prisms P_1, P_2 and P_3. Note that P_3 is obtained from P_1 by translation of one unit in the j-th coordinate direction. By the translation invariance of m_N we have $m_N(P_1) = m_N(P_3)$ and so

$$m_N\big(E(Q)\big) = m_N(P_2) + m_N(P_3) = m_N(P_2) + m_N(P_1) = m_N(Q) = 1.$$

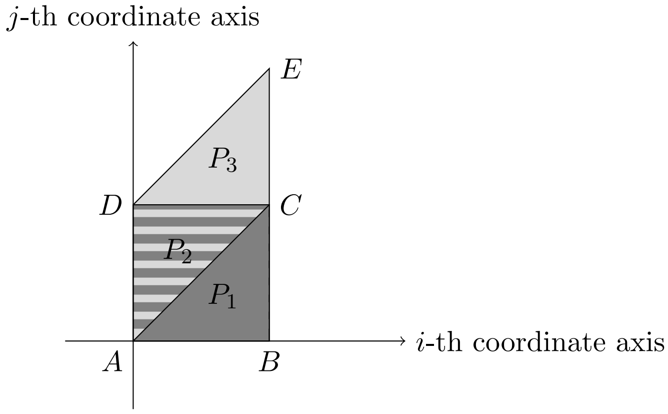

Figure 1: Effect on $(0,1]^N$ adding row i to row j.

(c) If T is a linear transformation on $\mathbb{R}^N$, prove that $m_N(T(Q)) = |\det T|$.

Solution: We use that $T = E_1E_2E_3\dots E_m$, where E_k are elementary matrices as in the previous parts. By applying the previous parts we get

$$m_N(T(Q)) = \prod_{k=1}^{m} |\det E_k| = |\det T| m_N(Q) = |\det T|.$$

(d) Let $U \subseteq \mathbb{R}^N$ be open and T a linear operator on $\mathbb{R}^N$. Show that $m_N(T(U)) = |\det T|$. Then argue why the same is true for any measurable set.

Hint: Use dyadic decomposition of open sets and Question 1.

Solution: Let U be an open set. According to lectures we can decompose U into a disjoint union

$$U = \bigcup_{k\in\mathbb{N}} Q_k,$$

where Q_k are diadic cubes. If that cube has edge of length 2^{-n_k}, then by the previous part and the translation invariance of the measure, $m(Q_k) = 2^{-n_kN}m(Q)$. Hence by the countable additivity of m

$$\begin{aligned} m(T(U)) = \sum_{k\in\mathbb{N}} m(T(Q_k)) &= \sum_{k\in\mathbb{N}} 2^{-n_kN} m(T(Q)) \\ &= \sum_{k\in\mathbb{N}} 2^{-n_kN} |\det T| m(Q) = |\det T| \sum_{k\in\mathbb{N}} m(Q_k) = |\det T| m(U). \end{aligned}$$

Next we use Question 1. If $A \subseteq \mathbb{R}^N$ is arbitrary, then

$$\begin{aligned} m^*(T(A)) &= \inf\{m(U) \colon T(A) \subseteq U,\, U \text{ open}\} \\ &= |\det T| \inf\{m(U) \colon A \subseteq T^{-1}(U),\, U \text{ open}\} = |\det T| m^*(A) \end{aligned}$$

Here we use that T is a linear bijective function and so $T^{-1}(U)$ is open if and only if U is open and hence $m^*(T(A)) = |\det T| m^*(A)$.

(e) Why is $m_N(T(U)) = 0$ if T is not invertible?

Solution: If T is not invertible, its image is in a lower dimensional subspace of $\mathbb{R}^N$ which has measure zero.

3. Let $\mu \colon \mathcal{A} \to [0,\infty]$ be a measure on X. Suppose that $f = (f_1,\dots,f_N) \colon X \to \mathbb{R}^N$.

(a) Suppose that $A_1, \dots, A_N \subseteq \mathbb{R}$. Show that $f^{-1}[A_1 \times \cdots \times A_N] = \bigcap_{k=1}^N f^{-1}[A_k]$.

Solution: Suppose that $x \in= f^{-1}[A_1 \times \cdots \times A_N]$. This is the case if and only if

$$f(x) = (f_1(x), f_2(x), \dots, f_N(x)) \in A_1 \times \cdots \times A_N.$$

Equivalently $f_1(x) \in A_1$, $f_2(x) \in A_2$,…,$f_N(x) \in A_N$, which means that $x \in \bigcap_{k=1}^N f^{-1}[A_k]$.

(b) If f is measurable, show that f_k is measurable for every $k = 1, \dots, N$.

Solution: Let $U \subset \mathbb{R}$ be open. We look at the inverse image of the open set

$$\mathbb{R} \times \cdots \times \mathbb{R} \times U \times \mathbb{R} \times \cdots \times \mathbb{R},$$

where U is in the k-th position. According to the previous part

$$f^{-1}[\mathbb{R} \times \cdots \times \mathbb{R} \times U \times \mathbb{R} \times \cdots \times \mathbb{R}] = X \cap \cdots \cap X \cap f_k^{-1}[U] \times X \cap \cdots \cap X = f_k^{-1}[U]$$

is measurable. Hence f_k is measurable.

(c) If f_k are measurable for every $k = 1, \dots, N$, show that f is measurable.

Solution: Suppose that $Q = I_1 \times \cdots \times I_N$ is a dyadic cube. Then by (a) and since f_k is measurable

$$f^{-1}[I_1 \times \cdots \times I_N] = \bigcap_{k=1}^N f^{-1}[I_k]$$

is measurable. If U is an open set we write $Q = \bigcup_{j \in \mathbb{N}} Q_j$, where Q_j are disjoint dyadic cubes. Hence

$$f^{-1}[U] = f^{-1}[\bigcup_{j \in \mathbb{N}} Q_j] = \bigcup_{j \in \mathbb{N}} f^{-1}[Q_j]$$

is measurable as claimed.

Extra questions for further practice

4. Consider a Lebesgue measurable set $A \subseteq \mathbb{R}^N$.

(a) Show that $m(A) = \sup\{m(K) \colon K \subseteq A,\ K \text{ compact}\}$.

Hint: First assume that A is bounded and approximate $S \cap A^c$ by open sets from outside for some compact set $S \supseteq A$. Then look at unbounded A.

Solution: Let A be a bounded Lebesgue measurable set. By the monotonicity of measures $m(K) \leq m(A)$ for all compact sets $K \subseteq A$. Hence we need to show that for every $\varepsilon > 0$ there exists a compact set $K \subset A$ with $m(K) > m(A) - \varepsilon$. Let S be a compact set such that $A \subseteq S$. By Question 1(a) there exists an open set U such that $m(U) \leq m(S \cap A^c) + \varepsilon$. We then set $K := S \cap U^c$. Then by the additivity of measures

$$\begin{aligned} m(K) = m(S \cap U^c) = m(S) - m(U) &> m(S) - m(S \cap A^c) - \varepsilon \\ &= m(S) - m(S) + m(A) - \varepsilon = m(A) - \varepsilon, \end{aligned}$$

so K is a compact set as required. Next we look at the case of unbounded A. We set $A_n := A \cap B(0, n)$, where $B(0, n)$ is the ball centred at zero with radius n. Then $m(A_n) \to m(A)$ as $n \to \infty$ by the monotonicity of measures. Given $\varepsilon > 0$ we choose $n \in \mathbb{N}$ such that $m(A_n) \geq m(A) - \varepsilon/2$. Then we choose a compact set K such that $m(K) > m(A_n) - \varepsilon/2$. Hence

$$m(K) > m(A_n) - \frac{\varepsilon}{2} \geq m(A) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = m(A) - \varepsilon.$$

This completes the proof.

(b) If A is Lebesgue measurable, show that there exists a Borel set C with $C \subseteq A$ and $m(A) = m(C)$.

Solution: According to (a), for every $n \in \mathbb{N}$ there exist compact sets $K_n \subseteq A$ such that $m(K_n) \to m(A)$. We set $C_1 := K_1$ and $C_{n+1} = C_n \cup K_{n+1}$. Then, $C_n \subseteq C_{n+1} \subseteq A$ for all $n \in \mathbb{N}$ and

$$m(A) \geq m(C_n) \geq m(K_n) \to m(A)$$

for all $n \in \mathbb{N}$. Set $C := \bigcup_{n=0}^{\infty} C_n$. Since C_n is compact C is a Borel set, and by the monotonicity of measures $m(C_n) \to m(C)$. Hence by the squeeze law $m(A) = m(C)$.

5. Let $\mu\colon \mathcal{A} \to [0, \infty]$ be a measure on the space X with $\mu(X) < \infty$. Suppose that $f\colon X \to \mathbb{R}$ is a μ-measurable function.

(a) Let $A_n := \{x \in X\colon |f(x)| > n\}$. Show that A_n is measurable and that $\mu(A_n) \to 0$ as $n \to \infty$.

Solution: The sets A_n are measurable because $|f|$ is measurable. Clearly $A_1 \supseteq A_2 \supseteq \dots$. Also, $\bigcap_{n=1}^{\infty} A_n = \emptyset$ because, if x lies in this intersection, then $|f(x)| \geq n$ for each n, and so $|f(x)| = \infty$, which is contrary to our hypothesis that f is a function with values in $\mathbb{R}$. Since $\mu(A_1) \leq \mu(X) < \infty$), we see that $\mu(A_n) \to m\big(\bigcap_{n=1}^{\infty} A_n\big) = \mu(\emptyset) = 0$ by the monotonicity properties of measures.

(b) Given $\varepsilon > 0$, show that there exist a simple function and a set $U \subseteq X$ such that $\varphi\colon X \to \mathbb{R}$ with $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in U$ and $\mu(X \setminus U) < \varepsilon$.
Hint: Look at non-negative functions first.

Solution: Suppose first that $f\colon X \to [0, \infty)$. By part (a), we can pick n so large that $\mu(A_n) < \epsilon$ and also $1/2^n < \epsilon$. Now let φ be the nth simple function constructed in lectures (see proof of Prop. 9.2 in printed notes). Then A_n is the set called $A_{n,n2^n}$ there. On all the other sets $A_{n,k}$ we have $|f(x) - \varphi(x)| < 1/2^n < \epsilon$. Hence, except on the set A_n (which has measure less than ϵ), $\varphi(x)$ differs from $f(x)$ by less than ϵ in modulus.
If $f\colon X \to \mathbb{R}$ takes positive and negative values, let $X^+ = \{x \in X\colon f(x) \geq 0\}$, let $X^- = \{x \in X\colon f(x) < 0\}$, and let α_n and β_n be the nth simple functions constructed in lectures (see proof of Prop. 9.2 in printed notes). Then A_n is the union of $\{x \in X\colon f(x)1_{X^+}(x) \geq n\}$ and $\{x \in X\colon -f(x)1_{X^-}(x) \geq n\}$. So outside A_n, $\alpha_n(x)$ and $\beta_n(x)$ are within $1/2^n$ of $f(x)1_{X^+}(x)$ and $-f(x)1_{X^-}(x)$, respectively. Let $\varphi = \alpha_n - \beta_n$. Then, outside A_n, $\varphi(x)$ differs from $f(x)$ by less than ϵ in modulus.

Challenge questions (optional)

6. Suppose $S \subseteq [0, 1]$ is a set which is not Lebesgue measurable.

(a) Show that the indicator function 1_S is not measurable.

Solution: Clearly $\{x \in [0, 1]\colon 1_S(x) > 0\} = S$ is not measurable by assumption on S. Hence 1_S is not measurable.

(b) Construct a function $f\colon [0, 1] \to \mathbb{R}$ which is not measurable, but the sets $\{x \in [0, 1]\colon f(x) = a\} = f^{-1}(\{a\})$ are measurable for all $a \in \mathbb{R}$.

Solution: Suppose that $S \subset [0, 1]$ is not Lebesgue measurable. Let $f(x) = x$ for $x \in S$, and let $f(x) = -x$ for $x \in [0, 1] \setminus S$. Then $f\colon [0, 1] \to \mathbb{R}$ is a function which does not take the same value twice. Hence, for each $\alpha \in \mathbb{R}$, the set $\{x \in [0, 1]\colon f(x) = \alpha\}$ is either empty or a singleton, and is therefore certainly Lebesgue measurable. But $\{x \in [0, 1]\colon f(x) > 0\}$ equals $S \setminus \{0\}$ or S, depending on whether $0 \in S$ or not. In either case, $\{x \in [0, 1]\colon f(x) > 0\}$ is not Lebesgue measurable, and so f is not Lebesgue measurable.