Material covered

(1) Properties and constructions of outer measures
(2) Construction of measures
(3) Properties of the Lebesgue measure.

Outcomes

After completing this tutorial you should

(1) be able to construct outer measures and measures
(2) know the main invariance properties of the Lebesgue measure
(3) be able to work with dyadic decomposition to prove properties of the Lebesgue measure.

Summary of essential material

The construction of measures from outer measures  Let $X$ be an arbitrary set. An outer measure on $X$ is a set function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ with the following properties:

(i) $\mu^*(\emptyset) = 0$

(ii) If $A, A_k \subseteq X$ with $A \subseteq \bigcup_{k=0}^{\infty} A_k$, then $\mu^*(A) \leq \sum_{k=0}^{\infty} \mu^*(A_k)$ (countable sub-additivity).

The sets in $\mathcal{A}$ are called measurable sets with respect to $\mu$ (or $\mu^*$). Note that an outer measure is always defined on the power set $\mathcal{P}(X)$, that is, for all subsets of $X$. Even if the sets $A_k$ are disjoint, we only require countable sub-additivity, not additivity. Carathéodory’s theorem states that

$$\mathcal{A} := \{ A \subseteq X : \mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \text{ for all } S \subseteq X \}$$

is a $\sigma$-algebra, and that $\mu := \mu^*|_A : \mathcal{A} \rightarrow [0, \infty]$ is a measure. We call $\mu$ the measure induced by the outer measure $\mu^*$.

The Lebesgue measure  The most important application of Carathéodory’s theorem for us is the construction of the Lebesgue measure. The Lebesgue measure in $\mathbb{R}^N$, that is, representing $N$-dimensional volume in $\mathbb{R}^N$, is the measure induced by the Lebesgue outer measure given by

$$m_N^*(A) := \inf \left\{ \sum_{k=0}^{\infty} \text{vol}_N(R_k) : R_k \text{ open rectangles, } A \subseteq \bigcup_{k=0}^{\infty} R_k \right\}.$$

The corresponding $\sigma$-algebra of $m_N^*$-measurable sets is called the Lebesgue $\sigma$-algebra $\mathcal{M}_N$. 

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Properties of the Lebesgue measure:

(i) The Lebesgue measure is a Borel measure. In particular, all open and closed sets are measurable, and all countable unions and intersections of open and closed sets are measurable.

(ii) The Lebesgue measure is the only translation invariant Borel measure on \( \mathbb{R}^N \) such that all bounded sets have finite measure.

(iii) The Lebesgue measure is regular, that is:

\[
m_N^*(A) = \inf \{ m_N(U) : A \subseteq U, \ U \text{ open} \} \quad \text{for every } A \subseteq \mathbb{R}^N.
\]

\[
m_N(A) = \sup \{ m_N(K) : A \supseteq K, \ K \text{ compact} \} \quad \text{for every measurable } A \subseteq \mathbb{R}^N.
\]

Dyadic decompositions Every non-empty open set in \( \mathbb{R}^N \) can be written as a disjoint union of countably many dyadic cubes. A dyadic cube is a cube of the form

\[
Q_{n,k} = \left\{ \frac{k}{2^n} + \left(0, \frac{1}{2^n}\right)^N : k \in \mathbb{Z}^N \right\}.
\]

Together with the regularity of the Lebesgue measure, dyadic decompositions are useful to prove many properties of the Lebesgue measure including uniqueness and invariance under rigid motion.

Questions to complete during the tutorial

1. Consider an arbitrary subset \( A \subseteq \mathbb{R}^N \). In lectures it was shown that

\[
m_N^*(A) = \inf \{ m_N(U) : A \subseteq U, \ U \text{ open} \} \quad \text{(outer regularity of } m_N^* \text{).}
\]

For simplicity, we throughout assume that \( m_N^*(A) < \infty \).

(a) Show that for every \( A \subseteq \mathbb{R}^N \) there exist open sets \( U_n \) so that

\[
m_N^*(A) = m_N\left( \bigcap_{n \in \mathbb{N}} U_n \right).
\]

In particular, the Borel set \( B = \bigcap_{n \in \mathbb{N}} U_n \) is such that \( A \subseteq B \) and \( m_N^*(A) = m_N(B) \).

**Solution:** According to (1) and the definition of an infimum, for every \( n \in \mathbb{N} \) there exists an open set \( U_n \subseteq \mathbb{R}^N \) such that \( A \subseteq U_n \) and

\[
m_N^*(A) \leq m_N(U_n) < m_N^*(A) + 1/n.
\]

In particular,

\[
A \subseteq B := \bigcap_{k \in \mathbb{N}} U_k \subseteq U_n
\]

for all \( n \in \mathbb{N} \). By the monotonicity of \( m_N^* \) and the choice of \( U_n \) we conclude that

\[
m_N^*(A) \leq m_N(B) \leq m_N(U_n) < m_N^*(A) + 1/n
\]

for all \( n \in \mathbb{N} \). Letting \( n \to \infty \) we deduce that \( m_N^*(A) \leq m_N(B) \leq m_N^*(A) \), that is, \( m_N^*(A) = m_N(B) \) as claimed. Since countable intersections of Borel sets are Borel sets, \( B \) is a Borel set. In particular \( m_N^*(A) = m_N(B) \).
2. Let $\Omega$ be the sample space obtained by doing an infinite sequence of coin tosses. Each such sequence can be represented in the form $(a_1, a_2, \ldots)$, where $a_k = 0$ or $1$. A zero means head and a one means tail for instance. Such sequences can be interpreted as a binary expansions of the number $a = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \in [0, 1]$.

There is one difficulty though: The association of a sequence $(a_1, a_2, \ldots)$ with the number $a$ is not one-to-one. As $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ every sequence ending with zeros gives the same $a$ as a sequence ending with ones. For instance,

$$(1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, \ldots)$$

$$(1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, \ldots)$$

represent the same $a$. For this reason we discard all sequences that only contain finitely many ones. What we discard is a set of probability zero since the probability of any single sequence occurring is zero. There are only countably many sequences with finitely many ones since this set can be written as the union of countably many finite sequences. In particular we discard the zero sequence $(0, 0, 0, \ldots)$. With that identification, the sample space becomes $\Omega = (0, 1]$.

(a) Set $A_k := \{(a_1, \ldots, a_k, \ldots) \in \Omega : a_k = 1\}$. Sketch $A_1$, $A_2$, $A_3$ as subsets of $(0, 1]$ and then describe the sets $A_k$ for general $k$. What is the probability of $A_k$, and how does it compare to the Lebesgue measure of $A_k$?

**Solution:** If $a_1 = 1$, then

$$a = \frac{1}{2} + \sum_{j=2}^{\infty} \frac{a_j}{2^j} > \frac{1}{2},$$

so $A_1 = \left(\frac{1}{2}, 1\right]$. There is a strict inequality since the sequence has to contain infinitely many ones. If $k = 2$, then

$$a = \frac{a_1}{2} + \frac{1}{4} + \sum_{j=3}^{\infty} \frac{a_j}{2^j}$$

so we have

$$A_2 = \left(\frac{1}{4}, \frac{1}{2}\right] \cup \left(\frac{3}{4}, 1\right]$$

since it is possible that $a_1 = 0$ or $a_1 = 1$. Again the left end is open because the sequence contains infinitely many ones by construction. Similarly,

$$A_3 = \left(\frac{1}{8}, \frac{1}{4}\right] \cup \left(\frac{3}{8}, \frac{1}{2}\right] \cup \left(\frac{5}{8}, \frac{3}{4}\right] \cup \left(\frac{7}{8}, 1\right].$$
The Lebesgue measure of each of the sets is clearly 1/2. A sketch of $A_1$, $A_2$ and $A_3$ is as follows:

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(b) Show that every dyadic interval $I_{n,j} = (j/2^n, (j + 1)/2^n]$, $j = 0, \ldots, 2^{n-1}$, can be written as a finite intersection of the sets $A_k$ and $A'_k$.

**Solution:** First note that $I_{n,j} \subseteq A_k$ or $I_{n,j} \subseteq A'_k$ for all $k = 1, \ldots, n$. Hence, for $k = 1, \ldots, n$ we let $B_{j,k} := A_k$ if $I_{n,j} \subseteq A_k$ and $B_{j,k} := A'_k$ if $I_{n,j} \subseteq A'_k$. Then $I_{n,j} = \cap_{k=1}^n B_{j,k}$.

(c) Argue why the probability measure in the above situation coincides with the Lebesgue measure on $(0, 1]$.

**Solution:** Every open subset $U$ of $(0, 1]$ can be written as a disjoint union of countably many intervals of the form $(j/2^n, (j + 1)/2^n]$ (dyadic decomposition). Using the additivity of the measure, the probability measure of $U$ is equal to the Lebesgue measure of $U$ and thus coincides. It follows that the probability measure is the same as the Lebesgue measure on the Borel sets. (Technically, we should show the regularity of the probability measure, but we leave that here. One can show that any Borel measure on an interval that is translation invariant, normalised and finite for bounded intervals coincides with the Lebesgue measure.)

3. The purpose of this question is to give an example of a set that is not Lebesgue measurable.

We define an equivalence relation on $\mathbb{R}^N$ by $x \sim y$ if $x - y \in \mathbb{Q}^N$. The equivalence class of $x$ is then given by $[x] := x + \mathbb{Q}^N$. Let now $B \subseteq \mathbb{R}^N$ be the unit ball. Construct a set $C$ as follows: For every equivalence class $[x]$ with $[x] \cap B \neq \emptyset$, choose precisely one $y \in B \cap [x]$ to be in $C$. Finally let $I := 2B \cap \mathbb{Q}^N$.

(a) Show that the union $\bigcup_{q \in I} (q + C)$ is disjoint.

**Solution:** Let $q_1, q_2 \in I$ and fix $x \in (q_1 + C) \cap (q_2 + C)$. Hence $x - q_1 \in C$ and $x - q_2 \in C$. Now $(x - q_1) - (x - q_2) = q_2 - q_1 \in \mathbb{Q}^N$ which means that $[x - q_1] = [x - q_2]$ by definition of the equivalence relation. By definition of $C$ we have $x - q_1 = x - q_2$ since we have chosen precisely one element from any given equivalence class to belong to $C$. This means that $q_1 = q_2$. Hence if $q_1 \neq q_2$, then $(q_1 + C) \cap (q_2 + C) = \emptyset$.

(b) Show that $B \subseteq \bigcup_{q \in I} (q + C) \subseteq 3B$

**Solution:** If $x \in B$, then clearly $B \cap [x] \neq \emptyset$ and so by construction of $C$ there exists $y \in C$ with $[x] = [y]$. However, this means that $x = q + y$ for some $q \in \mathbb{Q}^N$. Now $|q| = |x - y| \leq |x| + |y| < 2$ since $x, y \in B$. Hence $q \in I = 2B \cap \mathbb{Q}^N$ and so $x \in q + C$ for some $q \in I$.

For the second inclusion we let $q \in I$ and $x \in C$, then by definition of $I$ and $C$ we have $|q + x| \leq |q| + |x| < 2 + 1 = 3$. Hence $\bigcup_{q \in I} (q + C) \subseteq 3B$ as claimed.

(c) Show that $m_N^*(C) > 0$ and deduce that $C$ is not measurable.

**Solution:** Lebesgue measure is translation invariant and subadditive. Hence, using that $I$ is countable

$$0 < m_N^*(B) \leq m_N^* \left( \bigcup_{q \in I} (q + C) \right) \leq \sum_{q \in I} m_N^*(q + C) = \sum_{q \in I} m_N^*(C)$$
and therefore \( m_N^*(C) > 0 \). As \( I \) is a countably infinite set
\[
\sum_{q \in I} m_N^*(q + C) = \infty.
\]
If we assume that \( C \) is measurable, then \( q + C \) is also measurable and
\[
m_N^*(\bigcup_{q \in I} (q + C)) = \sum_{q \in I} m_N^*(q + C) \leq m_N^*(3B) < \infty
\]
because \( \bigcup_{q \in I} (q + C) \) is a disjoint union. As this is a contradiction we conclude that \( C \) is not measurable.

4. Let \((X, \mathcal{A}, \mu)\) be a measure space. For every \( A \subseteq X \) we define
\[
\mu^*(A) := \inf \left\{ \sum_{k=1}^{\infty} \mu(A_k) : A_k \in \mathcal{A}, k \in \mathbb{N}, A \subseteq \bigcup_{k=0}^{\infty} A_k \right\}
\]
(a) Show that \( \mu^* : \mathcal{P}(X) \rightarrow [0, \infty) \) is an outer measure. To do so proceed similarly as for the Lebesgue outer measure.

**Solution:** The proof is very similar to the proof that the Lebesgue outer measure is an outer measure.

If \( A = \emptyset \) we choose \( A_k = \emptyset \in \mathcal{A} \) for all \( k \in \mathbb{N} \). Then \( A \subseteq \bigcup_{k \in \mathbb{N}} A_k \). By definition of \( \mu^* \)
\[
0 \leq \mu^*(\emptyset) = \sum_{k=1}^{\infty} \mu(\emptyset) = 0 + 0 + 0 + \cdots = 0
\]
Hence \( \mu^*(\emptyset) = 0 \). To prove the countable sub-additivity let \( B, B_k \subseteq X \) so that \( B \subseteq \bigcup_{k \in \mathbb{N}} B_k \). By definition of \( \mu^* \) for every \( \varepsilon > 0 \) there exist sets \( A_{kj} \in \mathcal{A} \) such that
\[
\frac{\varepsilon}{2k+1} + \mu^*(B_k) > \sum_{j=0}^{\infty} \mu(A_{kj}).
\]
Moreover,
\[
B \subseteq \bigcup_{k \in \mathbb{N}} B_k \subseteq \bigcup_{k \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} A_{kj}.
\]
By definition of \( \mu^* \) we therefore find that
\[
\mu^*(B) \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \mu(A_{kj}) < \sum_{k=0}^{\infty} \left( \frac{\varepsilon}{2k+1} + \mu^*(B_k) \right) = \varepsilon + \sum_{k=0}^{\infty} \mu^*(B_k)
\]
As the above arguments work for every choice of \( \varepsilon > 0 \) we get
\[
\mu^*(B) \leq \varepsilon + \sum_{k=1}^{\infty} \mu^*(B_k)
\]
as claimed.

(b) Prove that \( \mu^*(A) \leq \mu(A) \) for all \( A \in \mathcal{A} \).

**Solution:** If \( A \in \mathcal{A} \), then we may choose \( A_0 := A \) and \( A_k = \emptyset \) for all \( k \geq 1 \). By definition of \( \mu^* \) we have
\[
\mu^*(A) \leq \sum_{k=0}^{\infty} \mu(A_k) = \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \cdots = \mu(A)
\]
as claimed.
(c) Prove that $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{A}$.

**Solution:** Taking into account the previous part we only need to show that $\mu^*(A) \geq \mu(A)$. Let $A \in \mathcal{A}$ and consider an arbitrary sequence of sets $A_k \in \mathcal{A}$ with $A \subseteq \bigcup_{k=0}^{\infty} A_k$. By the subadditivity of measures (see lectures)

$$\mu(A) \leq \sum_{k=0}^{\infty} \mu(A_k)$$

By definition of $\mu^*$ and properties of an infimum $\mu(A) \leq \mu^*(A)$.

**Extra questions for further practice**

5. Consider a Lebesgue measurable set $A \subseteq \mathbb{R}^N$.

   (a) Show that $m(A) = \sup\{m(K) : K \subseteq A, K \text{ compact}\}$.

   **Hint:** First assume that $A$ is bounded and approximate $S \cap A^c$ by open sets from outside for some compact set $S \supseteq A$. Then look at unbounded $A$.

   **Solution:** Let $A$ be a bounded Lebesgue measurable set. By the monotonicity of measures $m(K) \leq m(A)$ for all compact sets $K \subseteq A$. Hence we need to show that for every $\varepsilon > 0$ there exists a compact set $K \subset A$ with $m(K) > m(A) - \varepsilon$. Let $S$ be a compact set such that $A \subseteq S$. We know from lectures that there exists an open set $U$ such that $m(U) \leq m(S \cap A^c) + \varepsilon$. We then set $K := S \cap U^c$. Then by the additivity of measures

$$m(K) = m(S \cap U^c) = m(S) - m(U) > m(S) - m(S \cap A^c) - \varepsilon$$

$$= m(S) - m(S) + m(A) - \varepsilon = m(A) - \varepsilon,$$

so $K$ is a compact set as required. Next we look at the case of unbounded $A$. We set $A_n := A \cap B(0, n)$, where $B(0, n)$ is the ball centred at zero with radius $n$. Then $m(A_n) \to m(A)$ as $n \to \infty$ by the monotonicity of measures. Given $\varepsilon > 0$ we choose $n \in \mathbb{N}$ such that $m(A_n) \geq m(A) - \varepsilon/2$. Then we choose a compact set $K$ such that $m(K) > m(A_n) - \varepsilon/2$. Hence

$$m(K) > m(A_n) - \varepsilon/2 \geq m(A) - \varepsilon/2 - \varepsilon/2 = m(A) - \varepsilon.$$

This completes the proof.

(b) If $A$ is Lebesgue measurable, show that there exists a Borel set $C$ with $C \subseteq A$ and $m(A) = m(C)$.

**Solution:** According to (a), for every $n \in \mathbb{N}$ there exist compact sets $K_n \subseteq A$ such that $m(K_n) \to m(A)$. We set $C_1 := K_1$ and $C_{n+1} = C_n \cup K_{n+1}$. Then, $C_n \subseteq C_{n+1} \subseteq A$ for all $n \in \mathbb{N}$ and

$$m(A) \geq m(C_n) \geq m(K_n) \to m(A)$$

for all $n \in \mathbb{N}$. Set $C := \bigcup_{n=0}^{\infty} C_n$. Since $C_n$ is compact $C$ is a Borel set, and by the monotonicity of measures $m(C_n) \to m(C)$. Hence by the squeeze law $m(A) = m(C)$.

6. Show that the Lebesgue outer measure of an arbitrary set $A \subseteq \mathbb{R}^N$ is given by

$$m^*_N(A) = \inf \left\{ \sum_{k=0}^{\infty} m_N(Q_k) : Q_k \text{ disjoint dyadic cubes and } A \subseteq \bigcup_{k \in \mathbb{N}} Q_k \right\}.$$
For the proof use the outer regularity of the Lebesgue measure and the dyadic decomposition of open sets.

**Solution:** Let $Q_k$ be disjoint dyadic cubes such that $A \subseteq \bigcup_{k \in \mathbb{N}} Q_k$. From the subadditivity of the outer measure

$$m^*_N(A) \leq \sum_{k=0}^{\infty} m_N(Q_k)$$

and hence

$$m^*_N(A) \leq \inf \left\{ \sum_{k=0}^{\infty} m_N(Q_k) : Q_k \text{ disjoint dyadic cubes and } A \subseteq \bigcup_{k \in \mathbb{N}} Q_k \right\}.$$

of dyadic cubes $Q_k$. To prove the converse inequality we use the outer regularity of the measure, that is,

$$m^*(A) = \inf \{ m(U) : A \subseteq U, \text{ } U \text{ open} \}.$$ Given $\varepsilon > 0$ there exists an open set $U$ such that $A \subseteq U$ and $m(U) < m^*(A) + \varepsilon$. In lectures we showed that the open set $U$ can be written as a countable union of dyadic cubes $Q_k$. Hence by the countable additivity of the Lebesgue measure

$$m^*(A) + \varepsilon > m(U) = \sum_{k=0}^{\infty} m(Q_k).$$

We also know that $A \subseteq U = \bigcup_{k \in \mathbb{N}} Q_k$. As the above argument works for every choice of $\varepsilon > 0$ we get

$$m^*_N(A) \geq \inf \left\{ \sum_{k=0}^{\infty} m_N(Q_k) : Q_k \text{ disjoint dyadic cubes and } A \subseteq \bigcup_{k \in \mathbb{N}} Q_k \right\}$$

as required.