

Tutorial 3 (Week 4)

MATH3969: Measure Theory and Fourier Analysis (Advanced)

Semester 2, 2011

Web Page: <http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/>

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Questions to complete during the tutorial

1. Suppose that μ is a measure defined on the σ -algebra \mathcal{A} of subsets of X .
- (a) Show that for every measurable function $f: X \rightarrow \bar{\mathbb{R}}$ there exists a sequence (φ_n) of simple measurable functions with

$$|\varphi_1(x)| \leq |\varphi_2(x)| \leq |\varphi_3(x)| \leq \cdots \leq |f(x)|$$

for all $x \in X$ and $f(x) = \lim_{n \rightarrow \infty} \varphi_n(x)$ for all $x \in X$.

- (b) If we allow simple functions to take complex values, prove the assertions of the last exercise for complex valued measurable functions $f: X \rightarrow \mathbb{C}$.
2. (a) Let $A \subset \mathbb{R}^N$ be a non-empty subset and define $\text{dist}(x, A) =: \inf\{\|x - z\| : z \in A\}$. for every $x \in \mathbb{R}^N$. Show that

$$|\text{dist}(x, A) - \text{dist}(y, A)| \leq \|x - y\|$$

for all $x, y \in \mathbb{R}^N$. Conclude that the function $x \rightarrow \text{dist}(x, A)$ is continuous on \mathbb{R}^N .

- (b) Let $A, B \subseteq \mathbb{R}^N$ be non-empty closed sets with $A \cap B = \emptyset$. Using the distance function from (a) show that there is a continuous function $\phi: \mathbb{R}^N \rightarrow [0, 1]$ with $\phi(x) = 1$ for all $x \in A$ and $\phi(x) = 0$ for all $x \in B$.
- (c) Let $A \subset \mathbb{R}^N$ be a non-empty Lebesgue measurable set with $m(A) < \infty$ and 1_A the corresponding indicator function. Show that for every $\varepsilon > 0$ there exists a continuous function $\varphi: \mathbb{R}^N \rightarrow [0, 1]$ such that $\{x \in \mathbb{R}^N : \varphi(x) - 1_A(x) \neq 0\}$ has measure less than ε .
Hint: Use that $m(A) = \inf\{m(U) : A \subseteq U, U \text{ open}\} = \sup\{m(K) : K \subseteq A, K \text{ compact}\}$ and part (b).

3. Let $f: [a, b] \rightarrow \mathbb{R}$ be an increasing function.

- (a) Show that f is Lebesgue measurable on $[a, b]$.
- (b) Let

$$f(x+) = \lim_{y \rightarrow x, y > x} f(y) \quad \text{and} \quad f(x-) = \lim_{y \rightarrow x, y < x} f(y)$$

if $x \in [a, b]$ (writing $f(b+) = f(b)$ and $f(a-) = f(a)$). Show that these limits exist. Show that f has a discontinuity at x if and only if $f(x-) < f(x+)$.

- (c) For $n = 1, 2, \dots$, let

$$S_n = \{x \in [a, b] : f(x+) - f(x-) \geq \frac{1}{n}\}$$

Show that S_n is finite. Deduce that f has only a countable number of discontinuities (if any). The same conclusion holds if f is an increasing function $\mathbb{R} \rightarrow \mathbb{R}$.

Extra questions for further practice

The question below guides you through a proof of *Lusin's* theorem asserting that a measurable functions $A \rightarrow \mathbb{R}$ is “almost” continuous in the sense that there for every $\varepsilon > 0$ there exists a compact set $K \subset A$ such that $m(A \setminus K) < \varepsilon$ and $f: K \rightarrow \mathbb{R}$ is continuous.

4. Let $f: \mathbb{R}^N \rightarrow [0, \infty)$ be a measurable function and $A \subseteq \mathbb{R}^N$ measurable with $m(A) < \infty$. For $n, k \in \mathbb{N}$ define $B_{n,k} := [k/2^n, (k+1)/2^n)$ and

$$A_{n,k} := f^{-1}[B_{n,k}] \cap A = \{x \in A: k/2^n \leq f(x) < (k+1)/2^n\}.$$

Define $\varphi_n := \sum_{k=0}^{\infty} \frac{k}{2^n} 1_{A_{n,k}}$. Moreover, fix $\varepsilon > 0$.

- (a) Prove that $\varphi_n \rightarrow f$ uniformly on A .
- (b) Explain why there exist compact sets $K_{n,k} \subseteq A_{n,k}$ with $m(K_{n,k}) > m(A_{n,k}) - \varepsilon/2^{n+k+1}$.
- (c) Show that for every $n \in \mathbb{N}$ there exists $L_n \in \mathbb{N}$ such that

$$K_n := \bigcup_{k=0}^{L_n} K_{n,k}$$

satisfies $m(K_n) > m(A) - \varepsilon/2^n$.

- (d) Show that $\varphi_n: K_n \rightarrow \mathbb{R}$ is continuous.
- (e) Let $K := \bigcap_{n=1}^{\infty} K_n$. Show that $m(A \setminus K) < \varepsilon$ and that $f: K \rightarrow [0, \infty)$ is continuous.
- (f) Show that if f takes values in \mathbb{R} , then for every $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $m(A \setminus K) < \varepsilon$ and that $f: K \rightarrow \mathbb{R}$ is continuous.