The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 3 (Week 4)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
Lecturer: Daniel Daners

Questions to complete during the tutorial

1. Suppose that μ is a measure defined on the σ-algebra $\mathcal{A}$ of subsets of X.

(a) Show that for every measurable function $f\colon X \to \bar{\mathbb{R}}$ there exists a sequence (φ_n) of simple measurable functions with

$$|\varphi_1(x)| \leq |\varphi_2(x)| \leq |\varphi_3(x)| \leq \cdots \leq |f(x)|$$

for all $x \in X$ and $f(x) = \lim_{n\to\infty} \varphi_n(x)$ for all $x \in X$.

Solution: Let $X^+ = \{x \in X : f(x) \geq 0\}$ and let $X^- = \{x \in X : f(x) < 0\}$. Applying Proposition 2.3.4 to $f1_{X^+}$, there is a sequence (α_n) of simple functions such that for each $x \in X$,

$$0 \leq \alpha_1(x) \leq \alpha_2(x) \leq \cdots, \quad \text{and} \quad f(x)1_{X^+}(x) = \lim_{n\to\infty} \alpha_n(x).$$

Applying the same proposition to $-f1_{X^-}$, there is a sequence (β_n) of simple functions such that for each $x \in X$,

$$0 \leq \beta_1(x) \leq \beta_2(x) \leq \cdots, \quad \text{and} \quad -f(x)1_{X^-}(x) = \lim_{n\to\infty} \beta_n(x).$$

As $0 \leq \alpha_n(x) \leq f(x)1_{X^+}(x)$, we have $\alpha_n(x) = 0$ for all n and for all $x \in X^-$. Similarly, $\beta_n(x) = 0$ for all n and for all $x \in X^+$. Let $\varphi_n(x) = \alpha_n(x) - \beta_n(x)$. Suppose that $x \in X^+$. Then

$$|\varphi_n(x)| = |\alpha_n(x) - \beta_n(x)| = |\alpha_n(x) - 0| = \alpha_n(x),$$

which increases with n. Also, $\varphi_n(x) = \alpha_n(x)$ tends to $f(x)1_{X^+}(x) = f(x)$. Suppose now that $x \in X^-$. Then

$$|\varphi_n(x)| = |\alpha_n(x) - \beta_n(x)| = |0 - \beta_n(x)| = \beta_n(x),$$

which increases with n. Also, $\varphi_n(x) = -\beta_n(x)$ tends to $-(-f(x)1_{X^-}(x)) = f(x)$.

(b) If we allow simple functions to take complex values, prove the assertions of the last exercise for complex valued measurable functions $f\colon X \to \mathbb{C}$.

Solution: Write $f = u + iv$, where $u, v : X \to \mathbb{R}$ are measurable. Applying part (a) u and v separately, there is a sequence (α_n) of simple functions such that for each $x \in X$,

$$|\alpha_1(x)| \leq |\alpha_2(x)| \leq \cdots, \quad \text{and} \quad u(x) = \lim_{n\to\infty} \alpha_n(x);$$

there is also a sequence (β_n) of simple functions such that for each $x \in X$,

$$|\beta_1(x)| \leq |\beta_2(x)| \leq \cdots, \quad \text{and} \quad v(x) = \lim_{n\to\infty} \beta_n(x).$$

Then $\varphi_n(x) = \alpha_n(x) + i\beta_n(x)$ defines a complex-valued simple function φ_n. Clearly $\varphi_n(x) \to f(x)$ for each fixed $x \in X$. Also,

$$|\varphi_n(x)| = \sqrt{\alpha_n(x)^2 + \beta_n(x)^2}$$

increases with n because both $\alpha_n(x)^2$ and $\beta_n(x)^2$ do.

2. (a) Let $A \subset \mathbb{R}^N$ be a non-empty subset and define $\operatorname{dist}(x, A) =: \inf\{\|x - z\| : z \in A\}$. for every $x \in \mathbb{R}^N$. Show that

$$\big|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)\big| \leq \|x - y\|$$

for all $x, y \in \mathbb{R}^N$. Conclude that the function $x \to \operatorname{dist}(x, A)$ is continuous on $\mathbb{R}^N$.

Solution: For every $x, y \in \mathbb{R}^N$ and $z \in A$ we have

$$\operatorname{dist}(x, A) \leq \|x - z\| \leq \|x - y\| + \|y - z\|.$$

Taking the infimum over $z \in A$ on the right hand side we get

$$\operatorname{dist}(x, A) \leq \|x - y\| + \operatorname{dist}(y, A).$$

and therefore

$$\operatorname{dist}(x, A) - \operatorname{dist}(y, A) \leq \|x - y\|$$

for all $x, y \in \mathbb{R}^N$. Interchanging the roles of x and y we get

$$\operatorname{dist}(y, A) - \operatorname{dist}(x, A) \leq \|y - x\| = \|x - y\|.$$

Combining the two inequalities the required inequality follows. Continuity is obvious from the inequality since it clearly implies that $\operatorname{dist}(y, A) \to \operatorname{dist}(x, A)$ as $y \to x$.

(b) Let $A, B \subseteq \mathbb{R}^N$ be non-empty closed sets with $A \cap B = \emptyset$. Using the distance function from (a) show that there is a continuous function $\phi \colon \mathbb{R}^N \to [0, 1]$ with $\varphi(x) = 1$ for all $x \in A$ and $\varphi(x) = 0$ for all $x \in B$.

Solution: Since A, B are closed and disjoint we have that $\operatorname{dist}(x, B) + \operatorname{dist}(x, A) \neq 0$ for all $x \in \mathbb{R}^N$. Hence can set

$$\varphi(x) := \frac{\operatorname{dist}(x, B)}{\operatorname{dist}(x, B) + \operatorname{dist}(x, A)}$$

which is a continuous function into $[0, 1]$. If $x \in A$, then $\operatorname{dist}(x, A) = 0$ and therefore $\varphi(x) = 1$. On the other hand, if $x \in B$, then $\operatorname{dist}(x, B) = 0$ and therefore $\varphi(x) = 0$.

(c) Let $A \subset \mathbb{R}^N$ be a non-empty Lebesgue measurable set with $m(A) < \infty$ and 1_A the corresponding indicator function. Show that for every $\varepsilon > 0$ there exists a continuous function $\varphi \colon \mathbb{R}^N \to [0, 1]$ such that $\{x \in \mathbb{R}^N : \varphi(x) - 1_A(x) \neq 0\}$ has measure less than ε. *Hint:* Use that $m(A) = \inf\{m(U) : A \subseteq U,\ U \text{ open}\} = \sup\{m(K) : K \subseteq A,\ K \text{ compact}\}$ and part (b).

Solution: Fix $\varepsilon > 0$. From the hint there exists a compact set $K \subset A$ such that $m(K) > m(A) - \varepsilon/2$. There also exists an open set $U \supset A$ such that $m(U) < m(A) - \varepsilon/2$. Since K and U^c are closed sets there exists a continuous function $\varphi \colon \mathbb{R}^N \to [0, 1]$ with $\varphi(x) = 1$ for $x \in K$ and $\varphi(x) = 0$ for $x \in U^c$. Hence $\varphi(x) - 1_A(x) = 0$ for all $x \in K \cup U^c$ and possibly non-zero otherwise. Therefore

$$\begin{aligned} m\big(\{x \in \mathbb{R}^N : \varphi(x) - 1_A(x) \neq 0\}\big) &\leq m(U \setminus K) = m(U) - m(K) \\ &< m(A) - \frac{\varepsilon}{2} - \Big(m(A) - \frac{\varepsilon}{2}\Big) = \varepsilon. \end{aligned}$$

Thus φ is a continuous function as required.

3. Let $f \colon \mathbb{R} \to \mathbb{R}$ be an increasing function which is bounded on any bounded interval.

(a) Show that f is Borel measurable on $\mathbb{R}$.

Solution: According to lectures it is sufficient to show that

$$A_\alpha := \{x \in \mathbb{R} \colon f(x) > \alpha\}$$

is measurable for all $\alpha \in \mathbb{R}$. If α is given and $f(x) > \alpha$, then $f(y) \geq f(x) > \alpha$ for all $y \geq x$ by the monotonicity of f. Hence $[x, \infty) \subseteq A_\alpha$. We let

$$\beta := \inf\{x \in \mathbb{R} \colon f(x) > \alpha\} = \inf A_\alpha.$$

Then from the above reasoning $(\beta, \infty) \subseteq A_\alpha$, and $(-\infty, \beta) \cap A_\alpha = \emptyset$. There are two cases: If $f(\beta) \leq \alpha$, then $A_\alpha = (\beta, \infty)$. If $f(\beta) > \alpha$, then $[\beta, \infty) = A_\alpha$. The latter is possible if $\lim_{x \to \beta^-} f(x) \leq \alpha < f(\beta)$, that is, if f is discontinuous at β. In any case A_α is an interval and therefore Borel measurable as claimed.

(b) Let

$$f(x+) = \lim_{y \to x^+} f(y) \quad \text{and} \quad f(x-) = \lim_{y \to x^-} f(y)$$

if $x \in \mathbb{R}$. Show that these limits exist. Show that f has a discontinuity at x if and only if $f(x-) < f(x+)$.

Solution: Fix $x \in \mathbb{R}$ and set $\beta := \inf\{f(y) \colon y > x\}$. We show that $\beta = \lim_{y \to x^+} f(y)$. Fix $\varepsilon > 0$. By definition of β there exists $\delta > 0$ such that

$$\beta \leq f(x + \delta) < \beta + \epsilon$$

Since f is monotone we conclude that

$$\beta \leq f(y) \leq f(x + \delta) < \beta + \epsilon$$

for all $x < y < x + \delta$. Rearranging we have

$$|f(y) - \beta| < \varepsilon$$

whenever $0 < y - x < \delta$. By definition of a right limit the claim follows. The existence of the left limit is obtained similarly. From first year calculus we know that f is continuous if and only if left and right limits exist and are equal. If they are not equal, then due to the monotonicity of the function we must have $f(x-) < f(x+)$.

(c) For $n \in \mathbb{N}$ and any bounded interval, let

$$S_n = \{x \in [a, b] \colon f(x+) - f(x-) \geq \frac{1}{n}\}$$

Show that S_n is finite. Deduce that f has only a countable number of discontinuities (if any).

Solution: From the definition of S_n and since f is bounded on $[a, b]$ we have

$$\infty > f(b) - f(a) \geq \sum_{x \in S_n} f(x+) - f(x-) \geq \sum_{x \in S_n} \frac{1}{n} \geq 0$$

This is only possible if S_n is finite. From (b) we know that the set of discontinuities is given by

$$\{x \in [a, b] \colon f(x+) - f(x-) > 0\} = \bigcup_{n \in \mathbb{N}} S_n$$

Since a countable union of finite sets is countable, f has at most countably many discontinuities in $[a, b]$. The same is true for $\mathbb{R}$ since the above implies that on every interval $[k, k + 1]$, $k \in \mathbb{Z}$ the function f has at most countably many discontinuities. The same is the case on the countable union $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} [k, k + 1]$.

Extra questions for further practice

The question below guides you through a proof of *Lusin's* theorem asserting that a measurable functions $A \to \mathbb{R}$ is "almost" continuous in the sense that there for every $\varepsilon > 0$ there exists a compact set $K \subset A$ such that $m(A \setminus K) < \varepsilon$ and $f\colon K \to \mathbb{R}$ is continuous.

4. Let $f\colon \mathbb{R}^N \to [0,\infty)$ be a measurable function and $A \subseteq \mathbb{R}^N$ measurable with $m(A) < \infty$. For $n, k \in \mathbb{N}$ define $B_{n,k} := [k/2^n, (k+1)/2^n)$ and

$$A_{n,k} := f^{-1}[B_{n,k}] \cap A = \{x \in A\colon k/2^n \leq f(x) < (k+1)/2^n\}.$$

Define $\varphi_n := \sum_{k=0}^{\infty} \frac{k}{2^n} 1_{A_{n,k}}$. Moreover, fix $\varepsilon > 0$.

(a) Prove that $\varphi_n \to f$ uniformly on A.

Solution: Note that $A = \bigcup_{k=0}^{\infty}$ is a disjoint union. Hence φ_n is well defined since for every $x \in A$ only one term in the series is nonzero. Also, if $x \in A$, then there exists $n, k \in \mathbb{N}$ with $x \in A_{n,k}$ then

$$\frac{k}{2^n} = \varphi_n(x) \leq f(x) < \frac{k+1}{2^n} = \varphi_n(x) + \frac{1}{2^n}$$

and hence $|\varphi_n(x) - f(x)| < 1/2^n$ for all $n \in \mathbb{N}$. Hence $\varphi_n \to f$ uniformly on A.

(b) Explain why there exist compact sets $K_{n,k} \subseteq A_{n,k}$ with $m(K_{n,k}) > m(A_{n,k}) - \varepsilon/2^{n+k+1}$.

Solution: Recall that $m(A_{n,k} = \sup\{m(K)\colon K \subseteq A_{n,k},\ K \text{ compact}\}$. This means that there exists a compact set $K_{n,k} \subseteq A_{n,k}$ with $m(K_{n,k}) > m(A_{n,k}) - \varepsilon/2^{n+k+1}$.

(c) Show that for every $n \in \mathbb{N}$ there exists $L_n \in \mathbb{N}$ such that

$$K_n := \bigcup_{k=0}^{L_n} K_{n,k}$$

satisfies $m(K_n) > m(A) - \varepsilon/2^n$.

Solution: By choice of $K_{n,j}$ we have

$$\sum_{k=0}^{\infty} m(K_{n,k}) > \sum_{k=0}^{\infty}\Big(m(A_{n,k}) - \frac{\varepsilon}{2^{n+k+1}}\Big) = m(A) - \frac{\varepsilon}{2^{n+1}}\sum_{k=0}^{\infty}\frac{\varepsilon}{2^k} = m(A) - \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$. Hence there exists $L_n \in \mathbb{N}$ such that

$$m(K_n) := m\Big(\bigcup_{k=0}^{L_n} K_{n,k}\Big) = \sum_{k=0}^{L_n} m(K_{n,k}) > m(A) - \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$.

(d) Show that $\varphi_n\colon K_n \to \mathbb{R}$ is continuous.

Solution: Note that φ_n is constant and therefore continuous on $K_{n,k}$. Since disjoint compact sets are at a positive distance apart (by the continuity of the distance function and since continuous functions attain a minimum on every compact set), the function φ_n is continuous on the disjoint union $m(K_n) := m\Big(\bigcup_{k=0}^{L_n} K_{n,k}\Big)$ of finitely many compact sets.

(e) Let $K := \bigcap_{n=1}^{\infty} K_n$. Show that $m(A \setminus K) < \varepsilon$ and that $f\colon K \to [0,\infty)$ is continuous.

Solution: By definition of K_n we have

$$m(A \setminus K) = m\Big(A \cap \Big(\bigcap_{n=1}^{\infty} K_n\Big)^c\Big) = m\Big(A \cap \Big(\bigcup_{n=1}^{\infty} K_n^c\Big)\Big) < \sum_{n=1}^{\infty} m(A \setminus K_n) < \sum_{n=1}^{\infty}\frac{\varepsilon}{2^n} = \varepsilon.$$

By (d) φ_n is continuous on K and by (a) converges uniformly on K. Hence as the uniform limit of continuous functions is continuous, f is continuous on K.

(f) Show that if f takes values in $\mathbb{R}$, then for every $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $m(A \setminus K) < \varepsilon$ and that $f\colon K \to \mathbb{R}$ is continuous.

Solution: Just apply the above to the positive and negative parts of f and then take the union of the two (disjoint) compact sets. Then subtract the two functions.