Material covered

(1) approximation of simple functions;
(2) properties of measurable functions;

Outcomes

After completing this tutorial you should

(1) work with monotone approximations of measurable functions;
(2) be able to prove properties of measurable functions;

Summary of essential material

Let \((X, \mathcal{A}, \mu)\) be a measure space. and \(f : X \to \mathbb{K}^N\) a function. By definition, \(f\) is called measurable if \(f^{-1}[U] \in \mathcal{A}\) for every open set \(U \subseteq \mathbb{K}^N\). There are some equivalent statements:

- \(f\) is measurable if and only if \(f^{-1}[B] \in \mathcal{A}\) for every Borel set \(B \subseteq \mathbb{K}^N\). (We do not know whether \(f^{-1}[A] \in \mathcal{A}\) if \(A\) is only Lebesgue measurable!)
- \(f : X \to \mathbb{R}\) is measurable if and only if \(f^{-1}((a, \infty)) \in \mathcal{A}\) for every rational \(a \in \mathbb{R}\).

If \(\mu\) is a Borel measure, that is, \(\mathcal{A}\) contains all open sets, then every continuous function is measurable.

If \(A \subseteq X\) we define the indicator function \(1_A : X \to \mathbb{R}\) of \(A\) by

\[
1_A(x) := \begin{cases} 
1 & \text{if } x \in A; \\
0 & \text{if } x \notin A.
\end{cases}
\]

The indicator function is measurable if and only if \(A\) is measurable, that is, \(A \in \mathcal{A}\). Simple functions are functions with finite range. They can be written in the form

\[
\sum_{k=0}^{m} a_k 1_{A_k},
\]

where \(A_k\) are disjoint sets and \(1_{A_k}\) is the indicator function of \(A_k\). An simple function is measurable if and only if \(A_k \in \mathcal{A}\) for \(k = 0, \ldots, m\). Simple functions play an important role because every nonnegative measurable function \(f : X \to [0, \infty]\) can be approximated from below by measurable simple functions: There exist simple functions \(\varphi_n : X \to [0, \infty)\) such that \(\varphi_n \leq \varphi_{n+1} \leq f\) for all \(n \in \mathbb{N}\), and \(\varphi_n \to f\) pointwise.
Questions to complete during the tutorial

1. Let $\mu : A \to [0, \infty]$ be a measure on $X$. Suppose that $f = (f_1, \ldots, f_N) : X \to \mathbb{R}^N$.
   
   (a) Suppose that $A_1, \ldots, A_N \subseteq \mathbb{R}$. Show that $f^{-1}[A_1 \times \cdots \times A_N] = \bigcap_{k=1}^N f_k^{-1}[A_k]$.
   
   **Solution:** Suppose that $x \in f^{-1}[A_1 \times \cdots \times A_N]$. This is the case if and only if
   
   $$f(x) = (f_1(x), f_2(x), \ldots, f_N(x)) \in A_1 \times \cdots \times A_N.$$
   
   Equivalently $f_1(x) \in A_1, f_2(x) \in A_2, \ldots, f_N(x) \in A_N$, which means that $x \in \bigcap_{k=1}^N f_k^{-1}[A_k]$.

   (b) If $f$ is measurable, show that $f_k$ is measurable for every $k = 1, \ldots, N$.
   
   **Hint:** Use part (a) with $A_k = \mathbb{R}$ for all $k = 1, \ldots, N$, except for some $j$, where $A_j = U$ is an open set.

   **Solution:** Let $U \subseteq \mathbb{R}$ be open. We look at the inverse image of the open set
   
   $$\mathbb{R} \times \cdots \times \mathbb{R} \times U \times \mathbb{R} \times \cdots \times \mathbb{R},$$
   
   where $U$ is in the $k$-th position. According to the previous part
   
   $$f^{-1}[\mathbb{R} \times \cdots \times \mathbb{R} \times U \times \mathbb{R} \times \cdots \times \mathbb{R}] = X \cap \cdots \cap X \cap f_k^{-1}[U] \times X \cap \cdots \cap X = f_k^{-1}[U]$$
   
   is measurable. Hence $f_k$ is measurable.

   (c) If $f_k$ are measurable for every $k = 1, \ldots, N$, show that $f$ is measurable.
   
   **Hint:** Use part (a) and dyadic decompositions.

   **Solution:** Suppose that $Q = I_1 \times \cdots \times I_N$ is a dyadic cube. Then by (a) and since $f_k$ is measurable
   
   $$f^{-1}[I_1 \times \cdots \times I_N] = \bigcap_{k=1}^N f_k^{-1}[I_k]$$
   
   is measurable. If $U$ is an open set we write $U = \bigcup_{j \in \mathbb{N}} Q_j$, where $Q_j$ are disjoint dyadic cubes. Hence
   
   $$f^{-1}[U] = f^{-1}\left(\bigcup_{j \in \mathbb{N}} Q_j\right) = \bigcup_{j \in \mathbb{N}} f^{-1}[Q_j]$$
   
   is measurable as claimed.

2. Suppose $S \subseteq [0, 1]$ is a set which is not Lebesgue measurable.

   (a) Show that the indicator function $1_S$ is not measurable.

   **Solution:** Clearly $\{x \in [0, 1] : 1_S(x) > 0\} = S$ is not measurable by assumption on $S$. Hence $1_S$ is not measurable.

   (b) Construct a function $f : [0, 1] \to \mathbb{R}$ which is not measurable, but the sets
   
   $$f^{-1}\{a\} = \{x \in [0, 1] : f(x) = a\}$$
   
   are measurable for all $a \in \mathbb{R}$.

   **Solution:** Suppose that $S \subset [0, 1]$ is not Lebesgue measurable. Let $f(x) = x$ for $x \in S$, and let $f(x) = -x$ for $x \in [0, 1] \setminus S$. Then $f : [0, 1] \to \mathbb{R}$ is a function which does not take the same value twice. Hence, for each $a \in \mathbb{R}$, the set $\{x \in [0, 1] : f(x) = a\}$ is either empty or a singleton, and is therefore certainly Lebesgue measurable. But $\{x \in [0, 1] : f(x) > 0\}$ equals $S \setminus \{0\}$ or $S$, depending on whether $0 \in S$ or not. In either case, $\{x \in [0, 1] : f(x) > 0\}$ is not Lebesgue measurable, and so $f$ is not Lebesgue measurable.
3. Suppose that \( \mu \) is a measure defined on the \( \sigma \)-algebra \( \mathcal{A} \) of subsets of \( X \).

(a) Show that for every measurable function \( f : X \to [-\infty, \infty] \) there exists a sequence \( (\varphi_n) \) of simple measurable functions with

\[
|\varphi_1(x)| \leq |\varphi_2(x)| \leq |\varphi_3(x)| \leq \cdots \leq |f(x)|
\]

for all \( x \in X \) and \( f(x) = \lim_{n \to \infty} \varphi_n(x) \) for all \( x \in X \). (Apply the theorem for non-negative functions to the positive and negative parts of \( f \).)

**Solution:** Let \( X^+ = \{ x \in X : f(x) \geq 0 \} \) and \( X^- = \{ x \in X : f(x) < 0 \} \). Applying Proposition 2.3.4 to \( f1_{X^+} \), there is a sequence \( (\alpha_n) \) of simple functions such that for each \( x \in X \),

\[
0 \leq \alpha_1(x) \leq \alpha_2(x) \leq \cdots \quad \text{and} \quad f(x)1_{X^+}(x) = \lim_{n \to \infty} \alpha_n(x).
\]

Applying the same proposition to \( -f1_{X^-} \), there is a sequence \( (\beta_n) \) of simple functions such that for each \( x \in X \),

\[
0 \leq \beta_1(x) \leq \beta_2(x) \leq \cdots \quad \text{and} \quad -f(x)1_{X^-}(x) = \lim_{n \to \infty} \beta_n(x).
\]

As \( 0 \leq \alpha_n(x) \leq f(x)1_{X^+}(x) \), we have \( \alpha_n(x) = 0 \) for all \( n \) and for all \( x \in X^- \). Similarly, \( \beta_n(x) = 0 \) for all \( n \) and for all \( x \in X^+ \). Let \( \varphi_n(x) = \alpha_n(x) - \beta_n(x) \). Suppose that \( x \in X^+ \). Then

\[
|\varphi_n(x)| = |\alpha_n(x) - \beta_n(x)| = |\alpha_n(x) - 0| = \alpha_n(x),
\]

which increases with \( n \). Also, \( \varphi_n(x) = \alpha_n(x) \) tends to \( f(x)1_{X^+}(x) = f(x) \). Suppose now that \( x \in X^- \). Then

\[
|\varphi_n(x)| = |\alpha_n(x) - \beta_n(x)| = |0 - \beta_n(x)| = \beta_n(x),
\]

which increases with \( n \). Also, \( \varphi_n(x) = -\beta_n(x) \) tends to \( -(f(x)1_{X^-}(x)) = f(x) \).

(b) If we allow simple functions to take complex values, prove the assertions of the previous part for complex valued measurable functions \( f : X \to \mathbb{C} \).

**Solution:** Write \( f = u + iv \), where \( u, v : X \to \mathbb{R} \) are measurable. Applying part (a) \( u \) and \( v \) separately, there is a sequence \( (\alpha_n) \) of simple functions such that for each \( x \in X \),

\[
|\alpha_1(x)| \leq |\alpha_2(x)| \leq \cdots \quad \text{and} \quad u(x) = \lim_{n \to \infty} \alpha_n(x);
\]

there is also a sequence \( (\beta_n) \) of simple functions such that for each \( x \in X \),

\[
|\beta_1(x)| \leq |\beta_2(x)| \leq \cdots \quad \text{and} \quad v(x) = \lim_{n \to \infty} \beta_n(x).
\]

Then \( \varphi_n(x) = \alpha_n(x) + i\beta_n(x) \) defines a complex-valued simple function \( \varphi_n \). Clearly \( \varphi_n(x) \to f(x) \) for each fixed \( x \in X \). Also,

\[
|\varphi_n(x)| = \sqrt{\alpha_n(x)^2 + \beta_n(x)^2}
\]

increases with \( n \) because both \( \alpha_n(x)^2 \) and \( \beta_n(x)^2 \) do.
Extra questions for further practice

4. (a) Let $A \subset \mathbb{R}^N$ be a non-empty subset and define $\text{dist}(x, A) := \inf \{ ||x - z|| : z \in A \}$ for every $x \in \mathbb{R}^N$. Show that

$$\left| \text{dist}(x, A) - \text{dist}(y, A) \right| \leq ||x - y||$$

for all $x, y \in \mathbb{R}^N$. Conclude that the function $x \to \text{dist}(x, A)$ is continuous on $\mathbb{R}^N$.

**Solution:** For every $x, y \in \mathbb{R}^N$ and $z \in A$ we have

$$\text{dist}(x, A) \leq ||x - z|| \leq ||x - y|| + ||y - z||.$$ 

Taking the infimum over $z \in A$ on the right hand side we get

$$\text{dist}(x, A) \leq ||x - y|| \text{dist}(y, A),$$

and therefore

$$\text{dist}(x, A) - \text{dist}(y, A) \leq ||x - y||$$

for all $x, y \in \mathbb{R}^N$. Interchanging the roles of $x$ and $y$ we get

$$\text{dist}(y, A) - \text{dist}(x, A) \leq ||y - x|| = ||x - y||.$$

Combining the two inequalities the required inequality follows. Continuity is obvious from the inequality since it clearly implies that $\text{dist}(y, A) \to \text{dist}(x, A)$ as $y \to x$.

(b) Let $A, B \subseteq \mathbb{R}^N$ be non-empty closed sets with $A \cap B = \emptyset$. Using the distance function from (a) show that there is a continuous function $\varphi : \mathbb{R}^N \to [0, 1]$ with $\varphi(x) = 1$ for all $x \in A$ and $\varphi(x) = 0$ for all $x \in B$.

**Solution:** Since $A, B$ are closed and disjoint we have that $\text{dist}(x, B) + \text{dist}(x, A) \neq 0$ for all $x \in \mathbb{R}^N$. Hence can set

$$\varphi(x) := \frac{\text{dist}(x, B)}{\text{dist}(x, B) + \text{dist}(x, A)}$$

which is a continuous function into $[0, 1]$. If $x \in A$, then $\text{dist}(x, A) = 0$ and therefore $\varphi(x) = 1$. On the other hand, if $x \in B$, then $\text{dist}(x, B) = 0$ and therefore $\varphi(x) = 0$.

*(c)* Let $A \subset \mathbb{R}^N$ be a non-empty Lebesgue measurable set with $m(A) < \infty$ and $1_A$ the corresponding indicator function. Show that for every $\varepsilon > 0$ there exists a continuous function $\varphi : \mathbb{R}^N \to [0, 1]$ such that $m_N(\{x \in \mathbb{R}^N : \varphi(x) - 1_A(x) \neq 0 \}) < \varepsilon$.

**Hint:** Use that

$$m(A) = \inf \{ m(U) : A \subseteq U, U \text{ open} \} = \sup \{ m(K) : K \subseteq A, K \text{ compact} \}$$

and part (b).

**Solution:** Fix $\varepsilon > 0$. From the hint there exists a compact set $K \subseteq A$ such that $m(K) > m(A) - \varepsilon / 2$. There also exists an open set $U \supseteq A$ such that $m(U) < m(A) - \varepsilon / 2$.

Since $K$ and $U^c$ are closed sets there exists a continuous function $\varphi : \mathbb{R}^N \to [0, 1]$ with $\varphi(x) = 1$ for $x \in K$ and $\varphi(x) = 0$ for $x \in U^c$. Hence $\varphi(x) - 1_A(x) = 0$ for all $x \in K \cup U^c$ and possibly non-zero otherwise. Therefore

$$m(\{x \in \mathbb{R}^N : \varphi(x) - 1_A(x) \neq 0 \}) \leq m(U \setminus K) = m(U) - m(K)$$

$$< m(A) - \varepsilon / 2 - \left( m(A) - \varepsilon / 2 \right) = \varepsilon.$$ 

Thus $\varphi$ is a continuous function as required.
5. Let \( \mu : A \to [0, \infty] \) be a measure on the space \( X \) with \( \mu(X) < \infty \). Suppose that \( f : X \to \mathbb{R} \) is a \( \mu \)-measurable function.

(a) Let \( A_n := \{ x \in X : |f(x)| > n \} \). Show that \( A_n \) is measurable and that \( \mu(A_n) \to 0 \) as \( n \to \infty \).

**Solution:** The sets \( A_n \) are measurable because \( |f| \) is measurable. Clearly \( A_1 \supseteq A_2 \supseteq \ldots \). Also, \( \bigcap_{n=1}^{\infty} A_n = \emptyset \) because, if \( x \) lies in this intersection, then \( |f(x)| \geq n \) for each \( n \), and so \( |f(x)| = \infty \), which is contrary to our hypothesis that \( f \) is a function with values in \( \mathbb{R} \). Since \( \mu(A_1) \leq \mu(X) < \infty \), we see that \( \mu(A_n) \to m(\bigcap_{n=1}^{\infty} A_n) = \mu(\emptyset) = 0 \) by the monotonicity properties of measures.

*(b)* Given \( \epsilon > 0 \), show that there exist a simple function and a set \( U \subseteq X \) such that \( \varphi : X \to \mathbb{R} \) with \( |f(x) - \varphi(x)| < \epsilon \) for all \( x \in U \) and \( \mu(X \setminus U) < \epsilon \).

**Hint:** Look at non-negative functions first.

**Solution:** Suppose first that \( f : X \to [0, \infty) \). By part (a), we can pick \( n \) so large that \( \mu(A_n) < \epsilon \) and also \( 1/2^n < \epsilon \). Now let \( \varphi \) be the \( n \)-th simple function constructed in lectures. Then \( A_n \) is the set called \( A_{n,\epsilon} \), there. On all the other sets \( A_{n,k} \), we have
\[
|f(x) - \varphi(x)| < 1/2^n < \epsilon.
\]
Hence, except on the set \( A_n \) (which has measure less than \( \epsilon \)), \( \varphi(x) \) differs from \( f(x) \) by less than \( \epsilon \) in modulus.

If \( f : X \to \mathbb{R} \) takes positive and negative values, let \( X^+ = \{ x \in X : f(x) \geq 0 \} \), let \( X^- = \{ x \in X : f(x) < 0 \} \), and let \( \alpha_n \) and \( \beta_n \) be the \( n \)-th simple functions constructed in lectures. Then \( A_n \) is the union of \( \{ x \in X : f(x)1_{X^+}(x) \geq n \} \) and \( \{ x \in X : -f(x)1_{X^-}(x) \geq n \} \). So outside \( A_n \), \( \alpha_n(x) \) and \( \beta_n(x) \) are within \( 1/2^n \) of \( f(x)1_{X^+}(x) \) and \( -f(x)1_{X^-}(x) \), respectively. Let \( \varphi = \alpha_n - \beta_n \). Then, outside \( A_n \), \( \varphi(x) \) differs from \( f(x) \) by less than \( \epsilon \) in modulus.

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be an increasing function which is bounded on any bounded interval.

(a) Show that \( f \) is Borel measurable on \( \mathbb{R} \).

**Solution:** According to lectures it is sufficient to show that
\[
A_\alpha := \{ x \in \mathbb{R} : f(x) > \alpha \}
\]
is measurable for all \( \alpha \in \mathbb{R} \). If \( \alpha \) is given and \( f(x) > \alpha \), then \( f(y) \geq f(x) > \alpha \) for all \( y \geq x \) by the monotonicity of \( f \). Hence \( [x, \infty) \subseteq A_\alpha \). We let
\[
\beta := \inf \{ x \in \mathbb{R} : f(x) > \alpha \} = \inf A_\alpha.
\]
Then from the above reasoning \( (\beta, \infty) \subseteq A_\alpha \), and \( (-\infty, \beta) \cap A_\alpha = \emptyset \). There are two cases: If \( f(\beta) \leq \alpha \), then \( A_\alpha = (\beta, \infty) \). If \( f(\beta) > \alpha \), then \( [\beta, \infty) = A_\alpha \). The latter is possible if \( \lim_{x \to -\beta} f(x) \leq \alpha < f(\beta) \), that is, if \( f \) is discontinuous at \( \beta \). In any case \( A_\alpha \) is an interval and therefore Borel measurable as claimed.

(b) Let
\[
f(x+) = \lim_{y \to x^+} f(y) \quad \text{and} \quad f(x-) = \lim_{y \to x^-} f(y)
\]
if \( x \in \mathbb{R} \). Show that these limits exist. Show that \( f \) is discontinuous at \( x \) if and only if \( f(x-) < f(x+) \).

**Solution:** Fix \( x \in \mathbb{R} \) and set \( \beta := \inf \{ f(y) : y > x \} \). We show that \( \beta = \lim_{x \to x^+} f(y) \).

Fix \( \epsilon > 0 \). By definition of \( \beta \) there exists \( \delta > 0 \) such that
\[
\beta \leq f(x + \delta) < \beta + \epsilon
\]
Since \( f \) is monotone we conclude that

\[
\beta \leq f(y) \leq f(x + \delta) < \beta + \epsilon
\]

for all \( x < y < x + \delta \). Rearranging we have

\[
|f(y) - \beta| < \epsilon
\]

whenever \( 0 < y - x < \delta \). By definition of a right limit the claim follows. The existence of the left limit is obtained similarly. From first year calculus we know that \( f \) is continuous if and only if left and right limits exist and are equal. If they are not equal, then due to the monotonicity of the function we must have \( f(x^-) < f(x^+) \).

(c) For \( n \in \mathbb{N} \) and any bounded interval \([a, b] \), let

\[
S_n = \{ x \in [a, b] : f(x+) - f(x-) \geq \frac{1}{n} \}
\]

Show that \( S_n \) is finite and deduce that \( f \) has at most countably many discontinuities.

**Solution:** From the definition of \( S_n \) and since \( f \) is bounded on \([a, b] \) we have

\[
\infty > f(b) - f(a) \geq \sum_{x \in S_n} f(x+) - f(x-) \geq \sum_{x \in S_n} \frac{1}{n} \geq 0
\]

This is only possible if \( S_n \) is finite. From (b) we know that the set of discontinuities is given by

\[
\{ x \in [a, b] : f(x+) - f(x-) > 0 \} = \bigcup_{n \in \mathbb{N}} S_n
\]

Since a countable union of finite sets is countable, \( f \) has at most countably many discontinuities in \([a, b] \). The same is true for \( \mathbb{R} \) since the above implies that on every interval \([k, k+1], k \in \mathbb{Z} \) the function \( f \) has at most countably many discontinuities. The same is the case on the countable union \( \mathbb{R} = \bigcup_{k \in \mathbb{Z}} [k, k+1] \).

**Challenge questions (optional)**

7. The question below guides you through a proof of Lusin’s Theorem asserting that a measurable functions \( A \to \mathbb{R} \) is “almost” continuous in the sense that for every \( \epsilon > 0 \) there exists a compact set \( K \subset A \) such that \( m(A \setminus K) < \epsilon \) and \( f : K \to \mathbb{R} \) is continuous.

Let \( f : \mathbb{R}^N \to [0, \infty) \) be a measurable function and \( A \subseteq \mathbb{R}^N \) measurable with \( m(A) < \infty \). For \( n, k \in \mathbb{N} \) define \( J_{n,k} := [k/2^n, (k+1)/2^n) \) and

\[
A_{n,k} := f^{-1}[J_{n,k}] \cap A = \{ x \in A : k/2^n \leq f(x) < (k + 1)/2^n \}.
\]

Define \( \varphi_n := \sum_{k=0}^{\infty} \frac{1}{2^n} 1_{A_{n,k}} \). Moreover, fix \( \epsilon > 0 \).

(a) Prove that \( \varphi_n \to f \) uniformly on \( A \).

**Solution:** Note that \( A = \bigcup_{k=0}^{\infty} 1_{A_{n,k}} \) is a disjoint union. Hence \( \varphi_n \) is well defined since for every \( x \in A \) only one term in the series is nonzero. Also, if \( x \in A \), then there exists \( n, k \in \mathbb{N} \) with \( x \in A_{n,k} \) then

\[
k/2^n = \varphi_n(x) \leq f(x) < k + 1/2^n = \varphi_n(x) + 1/2^n
\]

and hence \( |\varphi_n(x) - f(x)| < 1/2^n \) for all \( n \in \mathbb{N} \). Hence \( \varphi_n \to f \) uniformly on \( A \).
(b) Explain why there exist compact sets $K_{n,k} \subseteq A_{n,k}$ with $m(K_{n,k}) > m(A_{n,k}) - \varepsilon/2^{n+k+1}$.

**Solution:** Recall that $m(A_{n,k}) = \sup\{m(K) : K \subseteq A_{n,k}, K \text{ compact}\}$. This means that there exists a compact set $K_{n,k} \subseteq A_{n,k}$ with $m(K_{n,k}) > m(A_{n,k}) - \varepsilon/2^{n+k+1}$.

(c) Show that for every $n \in \mathbb{N}$ there exists $L_n \in \mathbb{N}$ such that

$$K_n := \bigcup_{k=0}^{L_n} K_{n,k}$$

satisfies $m(K_n) > m(A) - \varepsilon/2^n$.

**Solution:** By choice of $K_{n,j}$ we have

$$\sum_{k=0}^{\infty} m(K_{n,k}) > \sum_{k=0}^{\infty} \left( m(A_{n,k}) - \frac{\varepsilon}{2^{n+k+1}} \right) = m(A) - \frac{\varepsilon}{2^{n+1}} \sum_{k=0}^{\infty} \frac{\varepsilon}{2^k} = m(A) - \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$. Hence there exists $L_n \in \mathbb{N}$ such that

$$m(K_n) := m\left( \bigcup_{k=0}^{L_n} K_{n,k} \right) = \sum_{k=0}^{L_n} m(K_{n,k}) > m(A) - \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$.

(d) Show that $\varphi_n : K_n \to \mathbb{R}$ is continuous.

**Solution:** Note that $\varphi_n$ is constant and therefore continuous on $K_{n,k}$. Since disjoint compact sets are at a positive distance apart (by the continuity of the distance function and since continuous functions attain a minimum on every compact set), the function $\varphi_n$ is continuous on the disjoint union $m(K_n) := m\left( \bigcup_{k=0}^{L_n} K_{n,k} \right)$ of finitely many compact sets.

(e) Let $K := \bigcap_{n=1}^{\infty} K_n$. Show that $m(A \setminus K) < \varepsilon$ and that $f : K \to [0, \infty)$ is continuous.

**Solution:** By definition of $K_n$ we have

$$m(A \setminus K) = m\left( A \cap \left( \bigcap_{n=1}^{\infty} K_n \right)^c \right) = m\left( A \cap \left( \bigcup_{n=1}^{\infty} K_n^c \right) \right) < \sum_{n=1}^{\infty} m(A \setminus K_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

By (d) $\varphi_n$ is continuous on $K$ and by (a) converges uniformly on $K$. Hence as the uniform limit of continuous functions is continuous, $f$ is continuous on $K$.

(f) Show that if $f$ takes values in $\mathbb{R}$, then for every $\varepsilon > 0$ there exists a compact set $K \subseteq A$ such that $m(A \setminus K) < \varepsilon$ and that $f : K \to \mathbb{R}$ is continuous.

**Solution:** Just apply the above to the positive and negative parts of $f$ and then take the union of the two (disjoint) compact sets. Then subtract the two functions.