THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 4 (Week 5)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) integrals of non-negative functions

(2) interchanging limits and integrals

(3) applications of the monotone convergence theorem

(4) simple substitution formulae

Outcomes

After completing this tutorial you should

(1) be able to prove elementary properties of integrals

(2) know how to apply the monotone convergence theorem

(3) have an appreciation of conditions allowing to interchange limits and integrals

Questions to complete during the tutorial

No tutorial due to quiz

Extra questions for further practice

1. (a) Show that if $U \subseteq \mathbb{R}^N$ is open and has Lebesgue measure zero, then $U = \emptyset$.

(b) Use (a) to show that if $f\colon \mathbb{R}^N \to [0,\infty)$ is continuous and if $\int_{\mathbb{R}^N} f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}^N$.

(c) Show that for an arbitrary measurable function $f\colon \mathbb{R}^N \to [0,\infty)$ we can have $\int_{\mathbb{R}^N} f(x)\,dx = 0$ without f being the zero function. What condition needs to be satisfied for the integral to be zero?

2. Think of a sequence (f_k) of nonnegative measurable functions on $\mathbb{R}$ such that $f(x) = \lim_{k\to\infty} f_k(x)$ for all $x \in \mathbb{R}$, but such that $\int_{\mathbb{R}} f_k(x)\,dx \not\to \int_{\mathbb{R}} f(x)\,dx$.

3. Suppose that μ is a measure defined on the σ-algebra $\mathcal{A}$ of subsets of X. Let $f\colon X \to [0,\infty)$ be a nonnegative measurable function with $\int_X f\,d\mu = C < \infty$. Show that for each $\alpha > 0$ we have

$$\mu\bigl(\{x \in X\colon f(x) \geq \alpha\}\bigr) \leq \frac{C}{\alpha}.$$

4. Let $t > 0$ be a fixed number. The function $f(x) := x^{t-1}e^{-x}$ is a non-negative measurable function on $(0,\infty)$. Hence we can define

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx.$$

The function $\Gamma\colon (0,\infty) \to \mathbb{R}$ is called the *Gamma function*.

(a) Sketch the graph of $y = f(x)$ for $x \in (0, \infty)$ and show that $\Gamma(t) < \infty$ for all $t > 0$.
Hint: Note that $x^{t-1}e^{-x} \leq x^{t-1}$ on $(0, 1]$ and that $x^{t-1}e^{-x} \leq C_t e^{-x/2}$ for suitable $C_t > 0$.

(b) Show that $\Gamma(1) = 1$.

(c) Show that $\Gamma(t + 1) = t\Gamma(t)$ for all $t > 0$. Deduce that $\Gamma(n + 1) = n!$ for all $n \in \mathbb{N}$.
Hint: Use integration by parts.

(d) For $k = 1, 2, \dots$, let

$$f_k(x) = \begin{cases} x^{t-1}\left(1 - \frac{x}{k}\right)^k & \text{if } 0 < x < k, \\ 0 & \text{if } k \leq x < \infty. \end{cases}$$

Show that $f_k(x) \to f(x)$ for every $x > 0$.

(e) Show that $f_k(x) \leq f_{k+1}(x)$ for all $x > 0$.
Hint: Use the arithmetic mean geometric mean inequality $a_1 a_2 \cdots a_m \leq (\frac{a_1+a_2+\cdots+a_m}{m})^m$ valid for $a_1, \dots, a_m \geq 0$.

(f) Use the monotone convergence theorem to derive the formula

$$\Gamma(t) = \lim_{k\to\infty} \frac{k!\, k^t}{t(t + 1)\dots(t + k)}$$

for all $t > 0$.

5. Let $f : \mathbb{R} \to \mathbb{C}$ be measurable.

(a) Show that the functions $x \to f(x - t)$ and $x \to f(-x)$ are measurable for every $t \in \mathbb{R}$.

(b) Prove that

$$\int_{-\infty}^{\infty} f(x - t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(-x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

Hint: First assume f is a nonnegative simple function on $\mathbb{R}$ and that $t \in \mathbb{R}$. Then do the general case.

(c) Prove that

$$\int_a^b f(x - t)\, dx = \int_{a-t}^{b-t} f(x)\, dx \qquad \text{and} \qquad \int_a^b f(-x)\, dx = \int_{-b}^{-a} f(x)\, dx.$$

Challenge questions (optional)

6. Generalise the Dominated Convergence Theorem as follows. Assume that $f_k : X \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is measurable for every $k \in \mathbb{N}$ and that $f_k \to f$ pointwise. Instead of assuming that there is a single integrable function $g : X \to [0, \infty]$ such that $|f_k(x)| \leq g(x)$ for all k and x, we assume that for each k there is an integrable function $g_k(x)$ such that

(i) $|f_k(x)| \leq g_k(x)$ for all k and all x,

(ii) $g(x) = \lim_{k\to\infty} g_k(x)$ exists for each $x \in X$,

(iii) $\int_X g_k\, d\mu \to \int_X g\, d\mu < \infty$ as $k \to \infty$.

Conclude that $\int_X f_k\, d\mu \to \int_X f\, d\mu$.

Hint: Use that if (a_k) and (b_k) are two sequences of real numbers, and if $a_k \to \ell \in \mathbb{R}$, then $\liminf_{k\to\infty}(a_k + b_k) = \ell + \liminf_{k\to\infty} b_k$.