Material covered

(1) integrals of non-negative functions
(2) interchanging limits and integrals
(3) applications of the monotone convergence theorem
(4) simple substitution formulae

Outcomes

After completing this tutorial you should

(1) be able to prove elementary properties of integrals
(2) know how to apply the monotone convergence theorem
(3) have an appreciation of conditions allowing to interchange limits and integrals

Summary of essential material

A simple function $\varphi : X \to Y$ is a function with finite range $\{a_0, a_1, \ldots, a_n\}$. Let $A_k := \varphi^{-1}\{\{a_k\}\}$. If $Y = \mathbb{R}^N$ or $\mathbb{C}^N$ (or some other vector space), then

$$\varphi = \sum_{k=0}^n a_k 1_{A_k} \quad (1)$$

is a linear combination of indicator functions for the disjoint sets $A_k$. If $(X, A, \mu)$ is a measure space and $Y$ a metric space (subset of $\mathbb{R}^N$ most of the time), then $\varphi$ is measurable if and only if $A_k \in A$ for all $k = 0, 1, \ldots, n$.

If $\varphi : X \to [0, \infty)$ is a simple measurable function of the form (1) with $A_k, k = 0, \ldots, n$, disjoint, we define the integral

$$\int_X \varphi \, d\mu := \sum_{k=0}^n a_k \mu(A_k). \quad (2)$$

If $\varphi, \psi : X \to [0, \infty)$ are measurable simple functions, then

- $\int_X (\varphi + \psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu$;
- $\int_X \alpha \varphi \, d\mu = \alpha \int_X \varphi \, d\mu$ for all $\alpha \geq 0$;
- If $0 \leq \varphi \leq \psi$, then $0 \leq \int_X \varphi \, d\mu \leq \int_X \psi \, d\mu$. 

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If \( f : X \to [0, \infty] \) is measurable we define
\[
\int_X f \, d\mu := \sup \left\{ \int_X \varphi \, d\mu : \varphi \text{ simple and measurable and } 0 \leq \varphi \leq f \right\}.
\] (3)

The two definitions coincide for simple functions as shown in lectures.

One of the key properties of the abstract Lebesgue integral is the monotone convergence theorem: Let \( f_n : X \to [0, \infty] \) be measurable functions such that \( 0 \leq f_n \leq f_{n+1} \) for all \( n \in \mathbb{N} \). Then
\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu.
\]

Note that due to the monotonicity \( \lim_{n \to \infty} f_n(x) \) exists in \([0, \infty]\) for all \( x \in X \). As a pointwise limit of measurable function this limit function is measurable.

A corollary is Fatou’s Lemma: Let \( f_n : X \to [0, \infty] \) be arbitrary measurable functions. Then
\[
\int_X \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.
\]

Questions to complete during the tutorial

1. Let \( f_n(x) := 2nx e^{-nx^2} \) for all \( x \geq 0 \) and \( n \in \mathbb{N} \). Then \( f_n(x) \to 0 \) for all \( x \geq 0 \) as \( n \to \infty \). Show that
\[
\int_0^\infty f_n(x) \, dx \neq 0.
\]

**Solution:** A simple substitution shows that
\[
\int_0^\infty 2nx e^{-nx^2} \, dx = -e^{-nx^2}\bigg|_0^\infty = 1
\]
for all \( n \in \mathbb{N} \). Hence, even though the integrand converges to zero pointwise, the limit of the integral is not zero.

2. Let \( \delta_a \) be the Dirac measure on a set \( X \) concentrated \( a \in X \). It is defined on the power set \( \mathcal{P}(X) \), so every set and every function is measurable.
   (a) Let \( \varphi = \sum_{k=0}^n a_k 1_{A_k} \) be a non-negative simple function. Find
   \[
   \int_X \varphi \, d\delta_a.
   \]
   **Solution:** Since \( \bigcup_{k=0}^n A_k = X \) is a disjoint union there exists precisely one \( m \in \{0, \ldots, n\} \) such that \( a \in A_m \). From the definition of the Dirac measure we have \( \delta_a(A_k) = 0 \) if \( k \neq m \) and \( \delta_a(A_m) = 1 \). Hence by definition of the integral of a simple function
   \[
   \int_X \varphi \, d\delta_a = \sum_{k=0}^n a_k \delta_a(A_k) = a_m = \varphi(a).
   \]

(b) Let \( f : X \to [0, \infty] \) be a function. From the definition of the integral of a non-negative function, determine \( \int_X f \, d\delta_a \).
Solution: Let $\varphi$ be a simple function such that $0 \leq \varphi \leq f$. Then by definition of the integral of $f$ as the supremum over all such $\varphi$ and part (a)

$$
\int_X \varphi \, d\delta_a = \varphi(a) \leq f(a),
$$

so

$$
\int_X f \, d\delta_a \leq f(a).
$$

To prove the reverse inequality we use the simple function $\varphi$ defined to be $f(a)$ at $x = a$ and zero otherwise. Then clearly $0 \leq \varphi \leq f$ and

$$
\int_X f \, d\delta_a \geq \int_X \varphi \, d\delta_a = f(a).
$$

Both inequalities together imply the required identity.

3. Let $(X, A, \mu)$ be a measure space and $g : X \to [0, \infty]$ a $\mu$-measurable function. For $A \in A$ define

$$
\nu(A) := \int_A g \, d\mu := \int_X 1_A g \, d\mu.
$$

(a) Prove that $\nu : A \to [0, \infty]$ is a measure.

*Hint:* Use the monotone convergence theorem to prove the countable additivity.

**Solution:** If $A = \emptyset$, then clearly

$$
\nu(\emptyset) := \int_\emptyset g \, d\mu = 0.
$$

For the countable additivity let $A_k \in A$, $k \in \mathbb{N}$, such that $A_k \cap A_j = \emptyset$ whenever $k \neq j$. Then

$$
g 1_{\bigcup_{k=0}^{\infty} A_k} = \sum_{k=0}^{\infty} g 1_{A_k}.
$$

Since $g \geq 0$ the monotone convergence theorem implies that

$$
\nu\left(\bigcup_{k=0}^{\infty} A_k\right) = \int_X \lim_{n \to \infty} \sum_{k=0}^{n} g 1_{A_k} \, d\mu = \lim_{n \to \infty} \sum_{k=0}^{n} \int_X g 1_{A_k} \, d\mu = \sum_{k=0}^{\infty} \int_{A_k} g \, d\mu = \sum_{k=0}^{\infty} \nu(A_k)
$$

as required.

(b) If $f \in L^1(X, \nu, [\nu])$, show that

$$
\int_X f \, d\nu = \int_X f g \, d\mu.
$$

*Hint:* Start with simple functions, then treat non-negative functions and then $\mathbb{R}$ and $\mathbb{C}$ valued functions.

**Solution:** If $f = \sum_{k=0}^{n} a_k 1_{A_k}$ is a non-negative simple function, then by definition of the integral

$$
\int_X f \, d\nu = \sum_{k=0}^{\infty} a_k \nu(A_k) = \sum_{k=0}^{\infty} a_k \int_{A_k} g \, d\mu = \int_{A_k} \sum_{k=0}^{\infty} a_k g \, d\mu = \int_X f g \, d\mu.
$$
Next let \( f : X \to [0, \infty] \) be measurable. From lectures we know that there exist simple measurable functions \( \varphi_n \) with \( \varphi_0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f \) and \( \varphi_n \to f \) pointwise. Hence by the above and the monotone convergence theorem

\[
\int_X f \, d\nu = \lim_{n \to \infty} \int_X \varphi_n \, d\nu = \lim_{n \to \infty} \int_X \varphi_n \, d\mu = \int_X f \, d\mu.
\]

Finally apply the above to positive and negative parts, and then to real and imaginary parts if \( f \) is real or complex valued.

4. Let \( \mu \) be the counting measure on \( \mathbb{N} \), that is,

\[
\mu(A) := \begin{cases} 
\text{card}(A) & \text{if } A \text{ is a finite set,} \\
\infty & \text{if } A \text{ is infinite.}
\end{cases}
\]

(a) Let \( f : \mathbb{N} \to [0, \infty) \), that is, \( f(k) := a_k \geq 0 \) for all \( k \in \mathbb{N} \). Use the monotone convergence theorem to show that

\[
\int_{\mathbb{N}} f \, d\mu = \sum_{k=0}^{\infty} a_k.
\]

**Solution:** Let \( f_n(k) := f(k) \) if \( 0 \leq k \leq n \) and \( f_n(k) := 0 \) for \( k > n \). Then \( f_n \) is a simple function attaining the finitely many values \( a_0, \ldots, a_n \) and 0. In particular,

\[
f_n = \sum_{k=0}^{n} f(k) \mathbb{1}(k) = \sum_{k=0}^{n} a_k \mathbb{1}(k).
\]

By definition of the integral of a simple function we have

\[
\int_{\mathbb{N}} f_n \, d\mu = \sum_{k=0}^{n} a_k \mu(\{k\}) = \sum_{k=0}^{n} a_k
\]

since \( \mu(\{k\}) = 1 \). From the definition of \( f_n \) we have \( f_n \leq f_{n+1} \) and \( \lim_{n \to \infty} f_n(k) = f(k) = a_k \) for all \( k \in \mathbb{N} \). By the monotone convergence theorem

\[
\int_{\mathbb{N}} f \, d\mu = \lim_{n \to \infty} \int_{\mathbb{N}} f_n \, d\mu = \lim_{n \to \infty} \sum_{k=0}^{n} a_k = \sum_{k=0}^{\infty} a_k.
\]

(b) Show that \( f : \mathbb{N} \to \mathbb{C} \) is integrable with respect to the counting measure if and only if \( \sum_{k=0}^{\infty} a_k \) is absolutely convergent. Here \( a_k := f(k) \).

**Solution:** Note that \( |f(k)| = |a_k| \) for all \( k \in \mathbb{N} \). Using part (a) integrable is equivalent to

\[
\int_{\mathbb{N}} |f| \, d\mu = \sum_{k=0}^{\infty} |a_k| < \infty.
\]

The latter is the definition of absolute convergence.

(c) Suppose that \( f : \mathbb{N} \to \mathbb{R} \) is integrable with respect to the counting measure. Use the definition of the general Lebesgue integral to show that

\[
\int_{\mathbb{N}} f \, d\mu = \sum_{k=0}^{\infty} a_k.
\]
where \( a_k \) := \( f(k) \).

**Solution:** We know that \( f^+(k) = \max\{f(k), 0\} = \max\{a_k, 0\} \) and \( f^-(k) = \max\{-f(k), 0\} = \max\{-a_k, 0\} \). Hence \( f(k) = f^+(k) - f^-(k) \) for all \( n \in \mathbb{N} \). By definition

\[
\int_{\mathbb{N}} f \, d\mu = \int_{\mathbb{N}} f^+ \, d\mu - \int_{\mathbb{N}} f^- \, d\mu = \sum_{k=0}^{\infty} f^+(k) - \sum_{k=0}^{\infty} f^-(k)
\]

\[
= \sum_{k=0}^{\infty} (f^+(k) - f^-(k)) = \sum_{k=0}^{\infty} a_k.
\]

A similar argument applies if we replace \( \mathbb{R} \) by \( \mathbb{C} \).

**Extra questions for further practice**

5. (a) Show that if \( U \subseteq \mathbb{R}^N \) is open and has Lebesgue measure zero, then \( U = \emptyset \).

**Solution:** Suppose that \( U \) is non-empty and let \( x \in U \). Since \( U \) is open there exists an open rectangle \( R \subseteq U \). We know that \( 0 < m(R) \leq m(U) \neq 0 \). By contrapositive the claim follows.

(b) Use (a) to show that if \( f : \mathbb{R}^N \to [0, \infty) \) is continuous and if \( \int_{\mathbb{R}^N} f(x) \, dx = 0 \), then \( f(x) = 0 \) for all \( x \in \mathbb{R}^N \).

**Solution:** Suppose that \( f(x) = 0 \) for all \( x \in \mathbb{R}^N \). Then obviously \( \int_{\mathbb{R}^N} f(x) \, dx = 0 \). We prove the converse by contrapositive and assume that \( f \) is continuous and non-zero. Then the sets

\[
U_n := f^{-1}[(1/n, \infty)] = \{x \in \mathbb{R}^N : f(x) > 1/n\}
\]

are open for every \( n \in \mathbb{N} \). Since

\[
\bigcup_{n \in \mathbb{N}} U_n = \{x \in \mathbb{R}^N : f(x) \neq 0\} \neq \emptyset
\]

there exists \( n \in \mathbb{N} \) such that \( U_n \neq \emptyset \) and so by (a) \( \mu(U_n) \neq 0 \). Hence

\[
\int_{\mathbb{R}^N} f(x) \, dx \geq \int_{\mathbb{R}^N} 1_{U_n} f(x) \, dx \geq \frac{1}{n} \int_{\mathbb{R}^N} 1_{U_n} \, dx = \frac{1}{n} \mu(U_n) > 0.
\]

Hence the integral is non-zero, proving the contrapositive.

(c) Show that for an arbitrary measurable function \( f : \mathbb{R}^N \to [0, \infty) \) it is possible to have \( \int_{\mathbb{R}^N} f(x) \, dx = 0 \) without \( f \) being the zero function. What condition needs to be satisfied for the integral to be zero?

**Solution:** An example of a non-zero function with zero integral is \( f = 1_A \), where \( A \) is a non-empty set of measure zero, for example a singleton \( A = \{a\} \). More generally, the integral is zero if \( \{x \in \mathbb{R}^N : f(x) \neq 0\} \) has zero measure.

We show that the above condition is also necessary for \( \int_{\mathbb{R}^N} f(x) \, dx = 0 \). We give a proof by contrapositive and assume that \( \{x \in \mathbb{R}^N : f(x) \neq 0\} \) has positive measure. Define \( U_n \) as above and note that

\[
\lim_{n \to \infty} m(U_n) = m\left(\bigcup_{n \in \mathbb{N}} U_n\right) = m\left(\{x \in \mathbb{R}^N : f(x) \neq 0\}\right) > 0.
\]

Hence there exists \( n \in \mathbb{N} \) such that \( m(U_n) > 0 \) and so

\[
\int_{\mathbb{R}^N} f(x) \, dx \geq \int_{\mathbb{R}^N} 1_{U_n} f(x) \, dx \geq \frac{1}{n} \int_{\mathbb{R}^N} 1_{U_n} \, dx = \frac{1}{n} \mu(U_n) > 0.
\]

Hence the integral is non-zero, proving the contrapositive.
6. Let \( f : \mathbb{R} \to \mathbb{C} \) be measurable.

(a) Show that the functions \( x \to f(x - t) \) and \( x \to f(-x) \) are measurable for every \( t \in \mathbb{R} \).

**Solution:** We first note that as a consequence of the definition, the Lebesgue measure is translation invariant, that is, \( m(t + A) = m(A) \) for every measurable set \( A \subseteq \mathbb{R} \) and \( t \in \mathbb{R} \), where \( t + A := \{ t + x : x \in A \} \). Similarly \( m(-A) = m(A) \) if we set \( -A := \{-x : x \in A \} \). It is also clear that \( t + A \) and \(-A\) are measurable if and only if \( A \) is measurable.

Let \( U \subseteq \mathbb{C} \) be open and consider \( A := F^{-1}[U] = \{ x \in \mathbb{R} : f(x) \in U \} \). As \( f \) is measurable \( A \) is measurable. Clearly, \( f(x - t) \in U \) if and only if \( x - t \in A \). Hence,

\[
\{ x \in \mathbb{R} : f(x - t) \in U \} = t + A
\]

which is measurable by the considerations above. Similarly, \( f(-x) \in U \) if and only if \(-x \in A \). Hence,

\[
\{ x \in \mathbb{R} : f(-x) \in U \} = -A,
\]

which is measurable as well.

(b) Prove that

\[
\int_{-\infty}^{\infty} f(x - t) \, dx = \int_{-\infty}^{\infty} f(x) \, dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(-x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx.
\]

**Hint:** First assume \( f \) is a nonnegative simple function on \( \mathbb{R} \) and that \( t \in \mathbb{R} \). Then do the general case.

**Solution:** If \( \varphi = \sum_{k=0}^{n} a_k 1_{A_k} \) is a non-negative simple function, then by the translation invariance of the Lebesgue measure

\[
\int_{-\infty}^{\infty} \varphi(x - t) \, dx = \sum_{k=0}^{m} a_k m(A_k - t) = \sum_{k=0}^{m} a_k m(A_k) = \int_{-\infty}^{\infty} \varphi(x) \, dx.
\]

Similarly

\[
\int_{-\infty}^{\infty} \varphi(-x) \, dx = \sum_{k=0}^{m} a_k m(-A_k) = \sum_{k=0}^{m} a_k m(A_k) = \int_{-\infty}^{\infty} \varphi(x) \, dx.
\]

Taking the supremum over all simple functions \( 0 \leq \varphi \leq f \), the assertions follow for non-negative measurable functions. For general \( f \) use the definition and apply the formula to positive and negative parts and then to real and imaginary parts.

(c) Prove that

\[
\int_{a}^{b} f(x - t) \, dx = \int_{a-t}^{b-t} f(x) \, dx \quad \text{and} \quad \int_{a}^{b} f(-x) \, dx = \int_{-b}^{-a} f(x) \, dx.
\]

**Solution:** If \( f \) is a measurable function on \((a, b)\) we simply set \( g(x) := f(x) \) if \( x \in (a, b) \) and \( g(x) = 0 \) otherwise. Then \( g \) is a measurable function and

\[
\int_{-\infty}^{\infty} g(x) \, dx = \int_{a}^{b} f(x) \, dx.
\]

Hence by the previous part

\[
\int_{a-t}^{b-t} f(x - t) \, dx = \int_{-\infty}^{\infty} g(x - t) \, dx = \int_{-\infty}^{\infty} g(x) \, dx = \int_{a}^{b} f(x) \, dx
\]

and similarly

\[
\int_{a}^{b} f(-x) \, dx = \int_{-\infty}^{\infty} g(-x) \, dx = \int_{-\infty}^{\infty} g(x) \, dx = \int_{a}^{b} f(x) \, dx
\]
Challenge questions (optional)

7. Generalise the Dominated Convergence Theorem as follows. Assume that \( f_k : X \to \mathbb{K} (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}) \) is measurable for every \( k \in \mathbb{N} \) and that \( f_k \to f \) pointwise. Instead of assuming that there is a single integrable function \( g : X \to [0, \infty] \) such that \( |f_k(x)| \leq g(x) \) for all \( k \) and \( x \), we assume that for each \( k \) there is an integrable function \( g_k(x) \) such that

(i) \( |f_k(x)| \leq g_k(x) \) for all \( k \) and all \( x \),

(ii) \( g(x) = \lim_{k \to \infty} g_k(x) \) exists for each \( x \in X \),

(iii) \( \int_X g_k \, d\mu \to \int_X g \, d\mu < \infty \) as \( k \to \infty \).

Conclude that \( \int_X f_k \, d\mu \to \int_X f \, d\mu \).

Hint: Use that if \((a_k)\) and \((b_k)\) are two sequences of real numbers, and if \( a_k \to \ell \in \mathbb{R} \), then

\[
\liminf_{k \to \infty} (a_k + b_k) = \ell + \liminf_{k \to \infty} b_k.
\]

Solution: As the pointwise limit of measurable functions, the function \( f \) is measurable. Passing to the limit in (i) we get \( |f(x)| \leq g(x) \) for all \( x \in X \), so \( f \) is integrable. Moreover,

\[
|f_k(x) - f(x)| \leq |f_k(x)| + |f(x)| \leq g_k(x) + g(x)
\]

for all \( x \in X \). Next note that by (ii)

\[
2g(x) = \lim_{k \to \infty} \left( g_k(x) + g_k(x) - |f_k(x) - f(x)| \right) = \liminf_{k \to \infty} \left( g(x) + g_k(x) - |f_k(x) - f(x)| \right)
\]

for all \( x \in X \). By the above \( g_k(x) + g(x) - |f_k(x) - f(x)| \geq 0 \), and so by Fatou’s Lemma and (iii)

\[
2 \int_X g \, d\mu = \int_X \liminf_{k \to \infty} \left( g(x) + g_k(x) - |f_k(x) - f(x)| \right) \, d\mu
\]

\[
= \liminf_{k \to \infty} \int_X g(x) + g_k(x) - |f_k(x) - f(x)| \, d\mu
\]

\[
= \liminf_{k \to \infty} \left( \int_X g(x) + g_k(x) \, d\mu - \int_X |f_k(x) - f(x)| \, d\mu \right)
\]

\[
= \lim_{k \to \infty} \left( \int_X g(x) + g_k(x) \, d\mu \right) + \liminf_{k \to \infty} \left( - \int_X |f_k(x) - f(x)| \, d\mu \right)
\]

\[
= 2 \int_X g \, d\mu - \limsup_{k \to \infty} \int_X |f_k(x) - f(x)| \, d\mu.
\]

If we rearrange the above we get

\[
\limsup_{k \to \infty} \int_X |f_k(x) - f(x)| \, d\mu \leq 0 \leq \liminf_{k \to \infty} \int_X |f_k(x) - f(x)| \, d\mu
\]

and hence

\[
\lim_{k \to \infty} \left| \int_X f_k \, d\mu - \int_X f \, d\mu \right| \leq \lim_{k \to \infty} \int_X |f_k(x) - f(x)| \, d\mu = 0.
\]

This concludes the proof.