THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 4 (Week 5)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) integrals of non-negative functions

(2) interchanging limits and integrals

(3) applications of the monotone convergence theorem

(4) simple substitution formulae

Outcomes

After completing this tutorial you should

(1) be able to prove elementary properties of integrals

(2) know how to apply the monotone convergence theorem

(3) have an appreciation of conditions allowing to interchange limits and integrals

Questions to complete during the tutorial

No tutorial due to quiz

Extra questions for further practice

1. (a) Show that if $U \subseteq \mathbb{R}^N$ is open and has Lebesgue measure zero, then $U = \emptyset$.

Solution: Suppose that U is non-empty and let $x \in U$. Since U is open there exists an open rectangle R with $R \subseteq U$. We know that $0 < m(R) \leq m(U) \neq 0$. By contrapositive the claim follows.

(b) Use (a) to show that if $f\colon \mathbb{R}^N \to [0,\infty)$ is continuous and if $\int_{\mathbb{R}^N} f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in \mathbb{R}^N$.

Solution: Suppose that $f(x) = 0$ for all $x \in \mathbb{R}^N$. Then obviously $\int_{\mathbb{R}^N} f(x)\,dx = 0$. We prove the converse by contrapositive and assume that f is continuous and non-zero. Then the sets

$$U_n := f^{-1}[(1/n,\infty)] = \{x \in \mathbb{R}^N : f(x) > 1/n\}$$

are open for every $n \in \mathbb{N}$. Since

$$\bigcup_{n\in\mathbb{N}} U_n = \{x \in \mathbb{R}^N : f(x) \neq 0\} \neq \emptyset$$

there exists $n \in \mathbb{N}$ such that $U_n \neq \emptyset$ and so by (a) $\mu(U_n) \neq 0$. Hence

$$\int_{\mathbb{R}^N} f(x)\,dx \geq \int_{\mathbb{R}^N} 1_{U_n} f(x)\,dx \geq \frac{1}{n}\int_{\mathbb{R}^N} 1_{U_n}\,dx = \frac{1}{n}\mu(U_n) > 0.$$

Hence the integral is non-zero, proving the contrapositive.

(c) Show that for an arbitrary measurable function $f\colon \mathbb{R}^N \to [0,\infty)$ we can have $\int_{\mathbb{R}^N} f(x)\,dx = 0$ without f being the zero function. What condition needs to be satisfied for the integral to be zero?

Solution: An example of a non-zero function with zero integral is $f = 1_A$, where A is a non-empty set of measure zero, for example a singleton $A = \{a\}$. More generally, the integral is zero if $\{x \in \mathbb{R}^N : f(x) \neq 0\}$ has zero measure.
We show that the above condition is also necessary for $\int_{\mathbb{R}^N} f(x)\,dx = 0$. We give a proof by contrapositive and assume that Define $\{x \in \mathbb{R}^N : f(x) \neq 0\}$ has positive measure. Define U_n as above and note that

$$\lim_{n\to\infty} m(U_n) = m\Bigl(\bigcup_{n\in\mathbb{N}} U_n\Bigr) = m\bigl(\{x \in \mathbb{R}^N : f(x) \neq 0\}\bigr) > 0.$$

Hence there exists $n \in \mathbb{N}$ such that $m(U_n) > 0$ and so

$$\int_{\mathbb{R}^N} f(x)\,dx \geq \int_{\mathbb{R}^N} 1_{U_n} f(x)\,dx \geq \frac{1}{n}\int_{\mathbb{R}^N} 1_{U_n}\,dx = \frac{1}{n}\mu(U_n) > 0.$$

Hence the integral is non-zero, proving the contrapositive.

2. Think of a sequence (f_k) of nonnegative measurable functions on $\mathbb{R}$ such that $f(x) = \lim_{k\to\infty} f_k(x)$ for all $x \in \mathbb{R}$, but such that $\int_{\mathbb{R}} f_k(x)\,dx \not\to \int_{\mathbb{R}} f(x)\,dx$.

Solution: Set $f_0 := 1_{[0,1]}$ and $f_k(x) := f_0(x-k)$. Then clearly $f_k(x) \to f(x) := 0$ for all $x \in \mathbb{R}$. However

$$\int_{\mathbb{R}} f_k(x)\,dx = 1 \not\to 0 = \int_{\mathbb{R}} 0\,dx$$

3. Suppose that μ is a measure defined on the σ-algebra $\mathcal{A}$ of subsets of X. Let $f\colon X \to [0,\infty)$ be a nonnegative measurable function with $\int_X f\,d\mu = C < \infty$. Show that for each $\alpha > 0$ we have

$$\mu\bigl(\{x \in X : f(x) \geq \alpha\}\bigr) \leq \frac{C}{\alpha}.$$

Solution: Let $A := \{x \in X : f(x) \geq \alpha\}$. Then

$$C = \int_X f\,d\mu \geq \int_X 1_A f\,d\mu \geq \alpha \int_X 1_A\,d\mu = \alpha\mu(A).$$

Hence the claim follows by dividing by $\alpha > 0$.

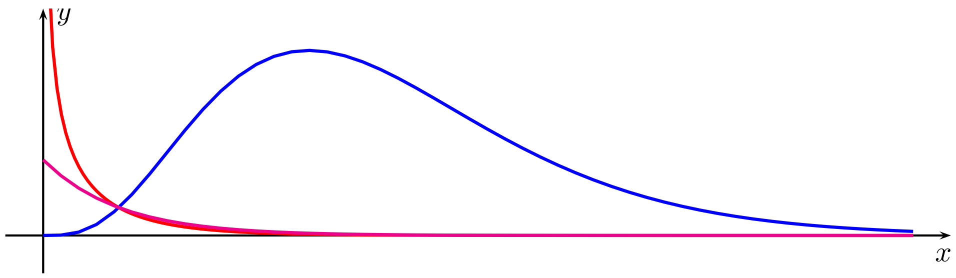
4. Let $t > 0$ be a fixed number. The function $f(x) := x^{t-1}e^{-x}$ is a non-negative measurable function on $(0,\infty)$. Hence we can define

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx.$$

The function $\Gamma\colon (0,\infty) \to \mathbb{R}$ is called the *Gamma function*.

(a) Sketch the graph of $y = f(x)$ for $x \in (0,\infty)$ and show that $\Gamma(t) < \infty$ for all $t > 0$.
Hint: Note that $x^{t-1}e^{-x} \leq x^{t-1}$ on $(0,1]$ and that $x^{t-1}e^{-x} \leq C_t e^{-x/2}$ for suitable $C_t > 0$.

Solution: Using the hint, the graph of f looks like for $t \in (0,1)$ (red), $t = 1$ (magenta) and $t > 1$ (blue):



It is clear that for $x \in (0,1]$ we have $x^{t-1}e^{-x} \leq x^{t-1}$ since $e^{-x \leq 1}$ for that range of x. For the other inequality note that $x^{t-1}e^{-x/2} \to 0$ as $x \to \infty$, so there exists $C_t > 0$ such that $x^{t-1}e^{-x/2} \leq C_t$ for all $x \geq 1$ and hence $x^{t-1}e^{-x} \leq C_t e^{-x/2}$. Now clearly

$$\int_0^1 x^{t-1}e^{-x}\,dx < \int_0^1 x^{t-1}\,dx = \frac{1}{t} < \infty \int_1^\infty x^{t-1}e^{-x}\,dx < \infty$$

for all $t > 0$. Also

$$\int_1^\infty x^{t-1}e^{-x}\,dx < C_t \int_1^\infty e^{-x/2}\,dx = 2C_t < \infty$$

for all $t > 0$. Adding the two together $\Gamma(t) < \infty$ for all $t > 0$.

(b) Show that $\Gamma(1) = 1$.

Solution: An explicit calculation yields

$$\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}\,dx = \int_0^\infty e^{-x}\,dx = 1.$$

(c) Show that $\Gamma(t+1) = t\Gamma(t)$ for all $t > 0$. Deduce that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.
Hint: Use integration by parts.

Solution: First note that $x^t e^{-x} \to 0$ as $x \to 0$ and $x \to \infty$. If we use integration by parts we get

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x}\,dx = -x^t e^{-x}\Big|_0^\infty + t\int_0^\infty x^{t-1}e^{-x}\,dx = t\Gamma(t)$$

for all $t > 0$. For the second part we use induction. We know already that $\Gamma(0+1) = \Gamma(1) = 1 = 0!$. Now assume that $\Gamma(n+1) = n!$. Hence, applying the above formula with $t = n+1$ we get, using the induction assumption,

$$\Gamma(n+2) = (n+1)\Gamma(n+1) = (n+1)n! = (n+1)!$$

as claimed.

(d) For $k = 1, 2, \dots$, let

$$f_k(x) = \begin{cases} x^{t-1}\bigl(1 - \frac{x}{k}\bigr)^k & \text{if } 0 < x < k, \\ 0 & \text{if } k \leq x < \infty. \end{cases}$$

Show that $f_k(x) \to f(x)$ for every $x > 0$.

Solution: We take the logarithm of the term involving k:

$$\log\Bigl(1 - \frac{x}{k}\Bigr)^k = k\log\Bigl(1 - \frac{x}{k}\Bigr) = -x\frac{\log\bigl(1 - \frac{x}{k}\bigr) - \log 1}{-\frac{x}{k}}.$$

If we let $k \to \infty$, then the above fraction converges to the derivative of log at one because it is the difference quotient for that derivative. Hence

$$\log\Bigl(1 - \frac{x}{k}\Bigr)^k \to x$$

and so $f_n(x) \to x^{t-1}e^{-x}$ as $k \to \infty$.

(e) Show that $f_k(x) \leq f_{k+1}(x)$ for all $x > 0$.
Hint: Use the arithmetic mean geometric mean inequality $a_1 a_2 \cdots a_m \leq (\frac{a_1 + a_2 + \cdots + a_m}{m})^m$ valid for $a_1, \dots, a_m \geq 0$.

Solution: Since $1 - \frac{x}{k} \geq 0$ for $0 \leq x < k$, we can apply the the algebraic mean geometric mean inequality with $k+1$ terms to conclude that

$$\Bigl(1 - \frac{x}{k}\Bigr)^k = \Bigl(1 - \frac{x}{k}\Bigr)^k \cdot 1 \leq \Bigl(\frac{k\bigl(1 + \frac{x}{n}\bigr) + 1}{k+1}\Bigr)^{k+1} = \Bigl(\frac{k+1+x}{k+1}\Bigr)^{k+1} = \Bigl(1 + \frac{x}{k+1}\Bigr)^{k+1}.$$

Since $f_k(x) = 0$ for $x \geq k$ we get $f_k(x) \leq f_{k+1}(x)$ for all $x > 0$ and all $k \in \mathbb{N}$.

(f) Use the monotone convergence theorem to derive the formula

$$\Gamma(t) = \lim_{k\to\infty} \frac{k!\,k^t}{t(t+1)\dots(t+k)}$$

for all $t > 0$.

Solution: From the previous parts and the monotone convergence theorem we have

$$\int_0^\infty x^{t-1}e^{-x}\,dx \int_0^\infty \lim_{k\to\infty} f_k(x)\,dx = \lim_{k\to\infty}\int_0^\infty f_k(x)\,dx = \lim_{k\to\infty}\int_0^k x^{t-1}\Bigl(1-\frac{x}{k}\Bigr)^k\,dx.$$

Hence we need to compute

$$I_k = \int_0^k x^{t-1}\Bigl(1-\frac{x}{k}\Bigr)^k\,dx.$$

We do that by making the substitution $u = x/k$. Then

$$I_k = k^t\int_0^1 u^{t-1}(1-u)^k\,du.$$

For arbitrary $\alpha > 0$ and $j \geq 1$ we have by integration by parts

$$\begin{aligned}\int_0^1 u^\alpha(1-u)^j\,du = \frac{u^{\alpha+1}}{\alpha+1}(1-u)^j\Big|_0^1 + \frac{j}{\alpha+1}\int_0^1 u^{\alpha+1}(1-u)^{j-1}\,dx \\ = \frac{j}{\alpha+1}\int_0^1 u^{\alpha+1}(1-u)^{j-1}\,dx.\end{aligned}$$

Using the above successively for $\alpha = t-1, t+1, t+2, \dots, t+k-1$ and $j = k, k-1, k-1, \dots, 0$ we get

$$\begin{aligned} I_k &= k^t\frac{1}{t-1}\int_0^1 u^{\alpha+1}(1-u)^k\,dx \\ &= k^t\frac{k}{t}\int_0^1 u^t(1-u)^{k-1}\,dx = k^t\frac{k(k-1)}{t(t+1)}\int_0^1 u^{t+1}(1-u)^{k-2}\,dx = \dots \\ &= k^t\frac{k(k-1)\cdots 2}{t(t+1)\cdots(t+k-1)}\int_0^1 u^{t+k-1}\,dx = \frac{k^t k!}{t(t+1)\cdots(t+k)}\end{aligned}$$

as claimed.

5. Let $f\colon \mathbb{R}\to\mathbb{C}$ be measurable.

(a) Show that the functions $x \to f(x-t)$ and $x \to f(-x)$ are measurable for every $t \in \mathbb{R}$.

Solution: We first note that as a consequence of the definition, the Lebesgue measure is translation invariant, that is, $m(t+A) = m(A)$ for every measurable set $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$, where $t + A := \{t + x\colon x \in A\}$. Similarly $m(-A) = m(A)$ if we set $-A := \{-x\colon x \in A\}$. It is also clear that $t + A$ and $-A$ are measurable if and only if A is measurable.

Fix $t \in \mathbb{R}$ and let $A := \{x \in \mathbb{R}\colon f(x) > a\}$. Therefore $f(x-t) > a$ if and only if $x-t \in A$, that is, $x \in t + A$. Since A is measurable also $t + A$ is measurable. Hence $x \mapsto f(x-t)$ is measurable. Similarly $f(-x) > a$ if and only if $-x \in A$, that is, $x \in -A$. Hence also $x \to f(-x)$ is measurable.

(b) Prove that

$$\int_{-\infty}^\infty f(x-t)\,dx = \int_{-\infty}^\infty f(x)\,dx \qquad\text{and}\qquad \int_{-\infty}^\infty f(-x)\,dx = \int_{-\infty}^\infty f(x)\,dx.$$

Hint: First assume f is a nonnegative simple function on $\mathbb{R}$ and that $t \in \mathbb{R}$. Then do the general case.

Solution: If $\varphi = \sum_{k=0}^{n} a_k 1_{A_k}$ is a non-negative simple function, then by the translation invariance of the Lebesgue measure

$$\int_{-\infty}^{\infty} \varphi(x-t)\,dx = \sum_{k=0}^{m} a_k m(A_k - t) = \sum_{k=0}^{m} a_k m(A_k) = \int_{-\infty}^{\infty} \varphi(x)\,dx.$$

Similarly

$$\int_{-\infty}^{\infty} \varphi(-x)\,dx = \sum_{k=0}^{m} a_k m(-A_k) = \sum_{k=0}^{m} a_k m(A_k) = \int_{-\infty}^{\infty} \varphi(x)\,dx.$$

Taking the supremum over all simple functions $0 \leq \varphi \leq f$, the assertions follow for non-negative measurable functions. For general f use the definition and apply the formula to positive and negative parts and then to real and imaginary parts.

(c) Prove that

$$\int_a^b f(x-t)\,dx = \int_{a-t}^{b-t} f(x)\,dx \qquad \text{and} \qquad \int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx.$$

Solution: If f is a measurable function on (a,b) we simply set $g(x) := f(x)$ if $x \in (a,b)$ and $g(x) = 0$ otherwise. Then g is a measurable function and

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_a^b f(x)\,dx.$$

Hence by the previous part

$$\int_{a-t}^{b-t} f(x-t)\,dx = \int_{-\infty}^{\infty} g(x-t)\,dx = \int_{-\infty}^{\infty} g(x)\,dx = \int_a^b f(x)\,dx$$

and similarly

$$\int_a^b f(-x)\,dx = \int_{-\infty}^{\infty} g(-x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx = \int_a^b f(x)\,dx$$

Challenge questions (optional)

6. Generalise the Dominated Convergence Theorem as follows. Assume that $f_k \colon X \to \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is measurable for every $k \in \mathbb{N}$ and that $f_k \to f$ pointwise. Instead of assuming that there is a single integrable function $g \colon X \to [0,\infty]$ such that $|f_k(x)| \leq g(x)$ for all k and x, we assume that for each k there is an integrable function $g_k(x)$ such that

(i) $|f_k(x)| \leq g_k(x)$ for all k and all x,

(ii) $g(x) = \lim_{k\to\infty} g_k(x)$ exists for each $x \in X$,

(iii) $\int_X g_k\,d\mu \to \int_X g\,d\mu < \infty$ as $k \to \infty$.

Conclude that $\int_X f_k\,d\mu \to \int_X f\,d\mu$.

Hint: Use that if (a_k) and (b_k) are two sequences of real numbers, and if $a_k \to \ell \in \mathbb{R}$, then $\liminf_{k\to\infty}(a_k + b_k) = \ell + \liminf_{k\to\infty} b_k$.

Solution: As the pointwise limit of measurable functions, the function f is measurable. Passing to the limit in (i) we get $|f(x)| \leq g(x)$ for all $x \in X$, so f is integrable. Moreover,

$$|f_k(x) - f(x)| \leq |f_k(x)| + |f(x)| \leq g_k(x) + g(x)$$

for all $x \in X$. Next note that by (ii)

$$2g(x) = \lim_{k\to\infty} \big(g_k(x) + g_k(x) - |f_k(x) - f(x)|\big) = \liminf_{k\to\infty} \big(g(x) + g_k(x) - |f_k(x) - f(x)|\big)$$

for all $x \in X$. By the above $g_k(x) + g(x) - |f_k(x) - f(x)| \geq 0$, and so by Fatou's Lemma and (iii)

$$\begin{aligned}
2\int_X g\,d\mu &= \int_X \liminf_{k\to\infty} \big(g(x) + g_k(x) - |f_k(x) - f(x)|\big)\,d\mu \\
&= \liminf_{k\to\infty} \int_X g(x) + g_k(x) - |f_k(x) - f(x)|\,d\mu \\
&= \liminf_{k\to\infty} \Big(\int_X g(x) + g_k(x)\,d\mu - \int_X |f_k(x) - f(x)|\,d\mu\Big) \\
&= \lim_{k\to\infty} \Big(\int_X g(x) + g_k(x)\,d\mu\Big) + \liminf_{k\to\infty} \Big(-\int_X |f_k(x) - f(x)|\,d\mu\Big) \\
&= 2\int_X g\,d\mu - \limsup_{k\to\infty} \int_X |f_k(x) - f(x)|\,d\mu.
\end{aligned}$$

If we rearrange the above we get

$$\limsup_{k\to\infty} \int_X |f_k(x) - f(x)|\,d\mu \leq 0 \leq \liminf_{k\to\infty} \int_X |f_k(x) - f(x)|\,d\mu$$

and hence

$$\lim_{k\to\infty} \Big| \int_X f_k\,d\mu - \int_X f(x)\,d\mu \Big| \leq \lim_{k\to\infty} \int_X |f_k(x) - f(x)|\,d\mu = 0.$$

This concludes the proof.