Material covered

(1) applications of the dominated convergence theorem
(2) applications of the monotone convergence theorem
(3) examples of measures and their construction
(4) integrals with respect to various measures
(5) simple distribution functions of measures

Outcomes

After completing this tutorial you should

(1) be able to apply the dominated convergence theorem in various situations
(2) be able to apply the monotone convergence theorem in a variety of contexts
(3) be able to determine simple distribution functions
(4) be more confident in applying fundamental techniques in measure theory like working with measurable functions, approximations and limit theorems.

Summary of essential material

Let \((X, \mathcal{A}, \mu)\) be a measure space. We call the function \(f : X \to \mathbb{K} (\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})\) integrable if

\[
\int_X |f| \, d\mu < \infty.
\]

The set of integrable functions forms a vector space and is denoted by \(L^1(X, \mathcal{A}, \mu)\). If it is clear from context what \(\mathcal{A}\) and \(\mu\) is we just write \(L^1(X, \mathcal{K})\) or \(L^1(X)\).

**Monotone Convergence Theorem** Let \((X, \mathcal{A}, \mu)\) be a measure space and \(f_n : X \to [0, \infty)\) be measurable functions such that \(0 \leq f_n \leq f_{n+1}\) for all \(n \in \mathbb{N}\). Then,

\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X \lim_{n \to \infty} f_n \, d\mu.
\]

**Dominated Convergence Theorem** Let \((X, \mathcal{A}, \mu)\) be a measure space and \(f_n : X \to \mathbb{K}\) measurable such that \(f_n \to f\) pointwise. Suppose that there exists an integrable function \(g : X \to [0, \infty]\) such that \(|f_n| \leq g\) for all \(n \in \mathbb{N}\). Then,

\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.
\]

We call the function \(g\) a dominating function. It is important that \(g\) is independent of \(n\) and integrable.
Questions to complete during the tutorial

1. Let \( f : \mathbb{R} \to \mathbb{C} \) be integrable. Show that \( g(t) = \int_{\mathbb{R}} f(x)e^{-2\pi i xt} \, dx \) is a bounded continuous function on \( \mathbb{R} \).

2. (a) Use the dominated convergence theorem to prove that
\[
\lim_{n \to \infty} n \int_0^1 \sqrt{x} e^{-n^2 x^2} \, dx = 0.
\]
(b) Compute
\[
\lim_{n \to \infty} n^2 \int_0^1 x e^{-n^2 x^2} \, dx
\]
and compare to taking the limit inside the integral.

3. Use the dominated convergence theorem to show that
\[
\lim_{n \to \infty} \int_{\mathbb{R}} (1 + \frac{x^2}{n})^{-\frac{n+1}{2}} \, dx = \int_{\mathbb{R}} e^{-\frac{x^2}{2}} \, dx.
\]
You may use that for every \( a \in \mathbb{R} \) we have \( (1 + \frac{a}{n})^n \to e^a \) as \( n \to \infty \).

4. Let \((X, \mathcal{A}, \mu)\) be a measure space and \( g : X \to \mathbb{R} \) measurable. Let \( f : \mathbb{R} \to \mathbb{R} \) be Borel measurable, that is, \( f^{-1}[B] \in \mathcal{B} \) for every \( B \in \mathcal{B} \), where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra in \( \mathbb{R} \).

(a) Prove that \( f \circ g : X \to \mathbb{R} \) is measurable. Explain why we need that \( f \) is Borel measurable and not just Lebesgue measurable.

(b) For \( B \in \mathcal{B} \) define \( \mu_g(B) = \mu(f^{-1}(B)) \). Show that \( \mu_g \) is a measure on \( \mathbb{R} \).

(c) If \( f : \mathbb{R} \to [0, \infty) \) is simple, prove that
\[
\int_X f \circ g \, d\mu = \int_{\mathbb{R}} f \, d\mu_g
\]
(1)

(d) Prove (1) for every non-negative Borel measurable function \( f : \mathbb{R} \to [0, \infty] \).

(e) Prove (1) for every Borel measurable function \( f : \mathbb{R} \to \mathbb{R} \).

(f) If \( g : \mathbb{R} \to \mathbb{R} \) is continuously differentiable, and \( \mu \) is the Lebesgue measure, identify the measure \( \mu_g \). Which method of integration does (1) correspond to in this case?

Extra questions for further practice

5. Let \((X, \mathcal{A}, \mu)\) be a measure space and \( u : X \to [0, M] \) a bounded measurable function. Assume that \( \mu(X) < \infty \) and let \( f : [0, \infty] \to [0, \mu(X)] \) be given by \( f(t) = \mu([u \geq t]) \), where \( [u \geq t] = \{ x \in X : u(x) \geq t \} \).

A partition \( P \) of \([0, M]\) is an ordered set \( \{t_0, t_1, \ldots, t_n\} \) such that \( 0 = t_0 < t_1 < t_2 < \cdots < t_n = M \). We define the size of \( P \) by \( ||P|| = \max_{k=1,\ldots,n} (t_k - t_{k-1}) \). We define the lower and upper Riemann sums (for the special case of a decreasing function) by
\[
L_P := \sum_{k=1}^n f(t_k)(t_k - t_{k-1}) \quad \text{and} \quad U_P := \sum_{k=1}^n f(t_{k-1})(t_k - t_{k-1})
\]
respectively. We define \( L := \sup_P L_P \) and \( U := \inf_P U_P \), where the supremum and infimum is over all possible partitions of \([0, M]\). We say \( f \) is Riemann integrable if \( L = U \).
(a) Given partitions $P_1$, $P_2$ and the refined partition $P = P_1 \cup P_2$ of $[0, M]$, show that
\[ U - \mu(X)\|P_1\| \leq L_{P_1} \leq L_P \leq U_P \leq U + \mu(X)\|P_2\| \]
and deduce that $f$ is Riemann integrable.

(b) Show that for every partition $P$ of $[0, M]$
\[ L_P \leq \int_X u \, d\mu \leq U_P \]
and deduce that
\[ \int_X f \, d\mu = \int_0^M \mu([u \geq t]) \, dt, \]
where the right hand side is a Riemann integral.

This essentially shows that a Lebesgue integral can be represented as a Riemann and is often referred to as Cavalieri’s principle or volume by horizontal slicing as done for instance to compute volumes of revolution by the disc or washer method.

(c) Let now $u : X \to [0, \infty]$ be measurable, but not necessarily bounded. Using the monotone convergence theorem, show that
\[ \int_X u \, d\mu = \int_0^\infty \mu([u \geq t]) \, dt, \]
with the right hand side being an improper Riemann integral.

6. Let $t > 0$ be a fixed number. The function $f(x) := x^{t-1}e^{-x}$ is a non-negative measurable function on $(0, \infty)$. Hence we can define
\[ \Gamma(t) = \int_0^\infty x^{t-1}e^{-x} \, dx. \]
The function $\Gamma : (0, \infty) \to \mathbb{R}$ is called the Gamma function.

(a) Sketch the graph $y = f(x)$ for $x \in (0, \infty)$ and show that $\Gamma(t) < \infty$ for all $t > 0$.

Hint: Note that $x^{t-1}e^{-x} \leq x^{t-1}$ on $(0, 1]$ and that $x^{t-1}e^{-x} \leq C_t e^{-x/2}$ for suitable $C_t > 0$.

(b) Show that $\Gamma(1) = 1$.

(c) Show that $\Gamma(t+1) = t\Gamma(t)$ for all $t > 0$. Deduce that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

Hint: Use integration by parts.

(d) For $k = 1, 2, \ldots$, let $f_k(x) = x^{t-1}\left(1 - \frac{x}{k}\right)_1^{0,k}$. Show that $f_k(x) \to f(x)$ for every $x > 0$.

(e) Show that $f_k(x) \leq f_{k+1}(x)$ for all $x > 0$.

Hint: Use the arithmetic mean geometric mean inequality $a_1a_2\cdots a_m \leq \left(\frac{a_1+a_2+\cdots+a_m}{m}\right)^m$ valid for $a_1, \ldots, a_m \geq 0$.

(f) Use the monotone convergence theorem to derive the formula
\[ \Gamma(t) = \lim_{k \to \infty} \frac{k! \, k^t}{t(t+1)\cdots(t+k)} \]
for all $t > 0$.

7. (a) Let $F(t) := 0$ for $t < 0$ and $F(t) = 1$ for $t \geq 0$. Determine the corresponding Lebesgue-Stieltjes measure.

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space for rolling a die and $P[E] := \text{card}(E)/6$ the probability of the event $E \subseteq \Omega$.

(i) What is the corresponding distribution function $F : \mathbb{R} \to [0, 1]$
What is the measure on \( \mathbb{R} \) corresponding to that distribution function.

8. Suppose that \((a, b) \subseteq \mathbb{R}\) is an interval (finite or infinite) and that \(f : (a, b) \to [0, \infty)\) is measurable.

(a) Show that \(f\) is integrable over \((a, b)\) if and only if

\[
\lim_{c \to b-} \int_a^c f(x) \, dx < \infty.
\]

Further show that in this case

\[
\int_a^b f(x) \, dx = \lim_{c \to b-} \int_a^c f(x) \, dx.
\]

A similar statement holds if we look at \(\lim_{c \to a+} \int_c^b f(x) \, dx\).

Hint: Use the monotone convergence theorem.

(In case of the Riemann integral, the latter is the definition of improper integrals. For the Lebesgue integral this provides a way to verify functions are integrable and compute the integrals)

(b) Determine for which \(a \in \mathbb{R}\) the following integrals are finite.

(i) \(\int_1^\infty \frac{1}{x^a} \, dx\)  
(ii) \(\int_0^1 \frac{1}{x^a} \, dx\)  
(iii) \(\int_0^\infty \frac{1}{x^a} \, dx\)  
(iv) \(\int_0^\infty e^{-ax} \, dx\)