The University of Sydney
School of Mathematics and Statistics

Tutorial 5 (Week 6)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) applications of the dominated convergence theorem

(2) applications of the monotone convergence theorem

(3) examples of measures and their construction

(4) integrals with respect to various measures

(5) simple distribution functions of measures

Outcomes

After completing this tutorial you should

(1) be able to apply the dominated convergence theorem in various situations

(2) be able ot apply the monotone convergence theorem in a variety of contexts

(3) be able to determine simple distribution functions

(4) be more confident in applying fundamental techniques in measure theory like working with measurable functions, approximations and limit theorems.

Questions to complete during the tutorial

1. Show that

$$\frac{n^{3/2}x^{3/2}}{1+n^2x^2} \leq \frac{1}{\sqrt{x}}$$

for all $x \in (0,1]$ and $n \in \mathbb{N}$. Conclude, using the Dominated Convergence Theorem, that

$$\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x^{3/2}}{1+n^2x^2}\,dx = 0.$$

Can you get that also by explicitly computing the integrals, or some other criterion?

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and $g\colon X \to [0,\infty]$ a μ-measurable function. For $A \in \mathcal{A}$ define

$$\nu(A) := \int_A g\,d\mu$$

(a) Prove that $\nu\colon \mathcal{A} \to [0,\infty]$ is a measure.
Hint: Use the monotone convergence theorem to prove the countable additivity.

(b) If $f \in \mathcal{L}^1(X,\nu,\mathbb{K})$, show that

$$\int_X f\,d\nu = \int_X fg\,d\mu.$$

Hint: Start with simple functions, then treat non-negative functions and then $\mathbb{R}$ and $\mathbb{C}$ valued functions.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and $g\colon X \to \mathbb{R}$ measurable. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Borel measurable, that is $f^{-1}[B] \in \mathcal{B}$ for every $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra in $\mathbb{R}$.

(a) Prove that $f \circ g\colon X \to \mathbb{R}$ is measurable. Explain why we need that f is Borel measurable and not just Lebesgue measurable.

(b) For $B \in \mathcal{B}$ define $\mu_g(B) := \mu\big(g^{-1}[B]\big)$. Show that μ_g is a measure on $\mathbb{R}$.

(c) If $f\colon \mathbb{R} \to [0, \infty)$ is simple, prove that

$$\int_X f \circ g \, d\mu = \int_{\mathbb{R}} f \, d\mu_g \tag{1}$$

(d) Prove (1) for every non-negative Borel measurable function $f\colon \mathbb{R} \to [0, \infty]$.

(e) Prove (1) for every Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$.

(f) If $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and μ the Lebesgue measure, identify the measure μ_g. Which method of integration does (1) correspond to in this case?

Extra questions for further practice

4. (a) Let $F(t) := 0$ for $t < 0$ and $F(t) = 1$ for $t \geq 0$. Determine the corresponding Lebesgue-Stieltjes measure.

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space for rolling a dice and $P[E] := \operatorname{card}(E)/6$ the probability of the event $E \subseteq \Omega$.

(i) What is the corresponding distribution function $F\colon \mathbb{R} \to [0, 1]$

(ii) What is the measure on $\mathbb{R}$ corresponding to that distribution function.

5. Suppose that $(a, b) \subseteq \mathbb{R}$ is an interval (finite or infinite) and that $f\colon (a, b) \to [0, \infty]$ is measurable.

(a) Show that f is integrable over (a, b) if and only if

$$\lim_{c \to b-} \int_a^c f(x)\, dx < \infty.$$

Further show that in this case

$$\int_a^b f(x)\, dx = \lim_{c \to b-} \int_a^c f(x)\, dx.$$

A similar statement holds if we look at $\lim_{c \to a+} \int_c^b f(x)\, dx$.
Hint: Use the monotone convergence theorem.
(In case of the Riemann integral, the latter is the *definition* of improper integrals. For the Lebesgue integral this provides a way to verify functions are integrable and compute the integrals)

(b) Determine for which $\alpha \in \mathbb{R}$ the following integrals are finite.

(i) $\displaystyle\int_1^\infty \frac{1}{x^\alpha}\, dx$ (ii) $\displaystyle\int_0^1 \frac{1}{x^\alpha}\, dx$ (iii) $\displaystyle\int_0^\infty \frac{1}{x^\alpha}\, dx$ (iv) $\displaystyle\int_0^\infty e^{-\alpha x}\, dx$

6. Let $\Gamma(t) := \int_0^\infty x^{t-1} e^{-x}\, dx$ be the Gamma function defined for $t > 0$.

(a) Prove that $\Gamma(t)$ is differentiable for all $t > 0$. Now set $g(x) := C_a x^{a/2-1}$ for $x \in (0, 1]$ and $g(x) := C_b e^{-x/2}$ if $x > 1$. Then $g \in \mathcal{L}^1((0, \infty), \mathbb{R})$ and

$$\left| \frac{\partial}{\partial t} x^{t-1} e^{-x} \right| \leq g(x)$$

for all $x > 0$ and all $t \in [a, b]$. Hence the theorem on the differentiation of parameter integrals applies, showing that $\Gamma\colon (0, \infty) \to \mathbb{R}$ is differentiable.

(b) Use the formula $\Gamma(t+1) = t\Gamma(t)$ to show that

$$\Gamma(t) = \frac{\Gamma(t+n)}{t(t+1)\cdots(t+n-1)}. \tag{2}$$

for all $n \in \mathbb{N}$.

(c) Define $\Gamma(t)$ by (2) for $-n < t < -n+1$, so that Γ is a function on $\mathbb{R} \setminus \{0, -1, -2, \dots\}$. Show that $\Gamma(t+1) = t\Gamma(t)$ on that domain.

(d) Show that Γ is differentiable on $\mathbb{R} \setminus \{0, -1, -2, \dots\}$.

7. Let μ be a measure defined on the σ-algebra $\mathcal{A}$ of subsets of X and let $f\colon X \to [0, \infty)$ be a measurable function. For $t > 0$ define $U_t := \{x \in X \colon f(x) > t\}$. The purpose of this question is to prove that

$$\int_X f\,d\mu = \int_0^\infty \mu(U_t)\,dt. \tag{3}$$

This is area/volume by horizontal slicing, often referred to as *Cavalieri's principle*. A special case is the disc method for computing volumes of revolution, where $\mu(U_t)$ corresponds to the area of the disc at level t.

(a) Let $0 < t_1 < t_2 < \cdots < t_n$ and $f = \sum_{k=0}^{n} t_k 1_{A_k}$ be a measurable simple function. Show that (3) holds.

(b) Prove (3) for every measurable function $f\colon X \to [0, \infty)$.