The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 5 (Week 6)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) applications of the dominated convergence theorem

(2) applications of the monotone convergence theorem

(3) examples of measures and their construction

(4) integrals with respect to various measures

(5) simple distribution functions of measures

Outcomes

After completing this tutorial you should

(1) be able to apply the dominated convergence theorem in various situations

(2) be able ot apply the monotone convergence theorem in a variety of contexts

(3) be able to determine simple distribution functions

(4) be more confident in applying fundamental techniques in measure theory like working with measurable functions, approximations and limit theorems.

Questions to complete during the tutorial

1. Show that

$$\frac{n^{3/2}x^{3/2}}{1+n^2x^2} \leq \frac{1}{\sqrt{x}}$$

for all $x \in (0,1]$ and $n \in \mathbb{N}$. Conclude, using the Dominated Convergence Theorem, that

$$\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x^{3/2}}{1+n^2x^2}\,dx = 0.$$

Can you get that also by explicitly computing the integrals, or some other criterion?

Solution: First note that

$$\frac{n^{3/2}x^{3/2}}{1+n^2x^2} \leq \frac{n^{3/2}x^{3/2}}{n^2x^2} = \frac{1}{\sqrt{nx}} \leq \frac{1}{\sqrt{x}}$$

as claimed. Also

$$0 \leq \frac{n^{3/2}x^{3/2}}{1+n^2x^2} \leq \frac{1}{\sqrt{nx}} \to 0.$$

Since

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_0^1 = 2 < \infty$$

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we can apply the dominated convergence theorem to conclude that

$$\lim_{n\to\infty}\int_0^1 \frac{n^{3/2}x^{3/2}}{1+n^2x^2}\,dx \int_0^1 \lim_{n\to\infty}\frac{n^{3/2}x^{3/2}}{1+n^2x^2}\,dx = 0.$$

Explicitly integrating seems hard or impossible. The sequence also does not converge monotonically or uniformly either, so the only easy option is the Dominated Convergence Theorem.

2. Let $(X,\mathcal{A},\mu)$ be a measure space and $g\colon X\to[0,\infty]$ a μ-measurable function. For $A\in\mathcal{A}$ define

$$\nu(A) := \int_A g\,d\mu$$

(a) Prove that $\nu\colon\mathcal{A}\to[0,\infty]$ is a measure.
Hint: Use the monotone convergence theorem to prove the countable additivity.

Solution: If $A=\emptyset$, then clearly

$$\nu(\emptyset) := \int_\emptyset g\,d\mu = 0$$

For the countable additivity let $A_k\in\mathcal{A}$, $k\in\mathbb{N}$, such that $A_k\cap A_j=\emptyset$ whenever $k\neq j$. Then

$$g1_{\bigcup_{k=0}^\infty A_k} = \sum_{k=0}^\infty g1_{A_k}.$$

Since $g\geq 0$ the monotone convergence theorem implies that

$$\nu\Big(\bigcup_{k=0}^\infty A_k\Big) = \int_X \lim_{n\to\infty}\sum_{k=0}^n g1_{A_k}\,d\mu = \lim_{n\to\infty}\sum_{k=0}^n\int_X g1_{A_k}\,d\mu = \sum_{k=0}^\infty\int_{A_k} g\,d\mu = \sum_{k=0}^\infty \nu(A_k)$$

as required.

(b) If $f\in\mathcal{L}^1(X,\nu,\mathbb{K})$, show that

$$\int_X f\,d\nu = \int_X fg\,d\mu.$$

Hint: Start with simple functions, then treat non-negative functions and then $\mathbb{R}$ and $\mathbb{C}$ valued functions.

Solution: If $f=\sum_{k=0}^n a_k 1_{A_k}$ is a non-negative simple function, then by definition of the integral

$$\int_X f\,d\nu = \sum_{k=0}^\infty a_k\nu(A_k) = \sum_{k=0}^\infty a_k\int_{A_k} g\,d\mu = \int_{A_k}\sum_{k=0}^\infty a_k g\,d\mu = \int_X fg\,d\mu.$$

Next let $f\colon X\to[0,\infty]$ be measurable. From lectures we know that there exist simple measurable functions φ_n with $\varphi_0\leq\varphi_1\leq\varphi_2\leq\cdots\leq f$ and $\varphi_n\to f$ pointwise. Hence by the above and the monotone convergence theorem

$$\int_X f\,d\nu = \lim_{n\to\infty}\int_X \varphi_n\,d\nu = \lim_{n\to\infty}\int_X \varphi_n g\,d\mu = \int_X fg\,d\mu.$$

Finally apply the above to positive and negative parts, and then to real and imaginary parts if f is real or complex valued.

3. Let $(X,\mathcal{A},\mu)$ be a measure space and $g\colon X\to\mathbb{R}$ measurable. Let $f\colon\mathbb{R}\to\mathbb{R}$ be Borel measurable, that is $f^{-1}[B]\in\mathcal{B}$ for every $B\in\mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra in $\mathbb{R}$.

(a) Prove that $f \circ g \colon X \to \mathbb{R}$ is measurable. Explain why we need that f is Borel measurable and not just Lebesgue measurable.

Solution: Let $U \subseteq \mathbb{R}$ be open. Since f is Borel measurable $f^{-1}[U] \subseteq \mathbb{R}$ is a Borel set. As g is measurable we get

$$(f \circ g)^{-1}[U] = g^{-1}\big[f^{-1}[U]\big] \in \mathcal{A}.$$

Here we use the fact from lectures that for a measurable function $g^{-1}[B] \in \mathcal{A}$ for every Borel set. If f is measurable but not Borel measurable, then $B := f^{-1}[U]$ is not necessarily a Borel set, and therefore $g^{-1}[B]$ not necessarily in $\mathcal{A}$.

(b) For $B \in \mathcal{B}$ define $\mu_g(B) := \mu\big(g^{-1}[B]\big)$. Show that μ_g is a measure on $\mathbb{R}$.

Solution: Since g is Borel measurable $g^{-1}[B]$ is a Borel set and therefore μ_g is well defined. Now $g^{-1}[\emptyset] = \emptyset$, and therefore

$$\mu_g(\emptyset) = \mu\big(g^{-1}[\emptyset]\big) = \mu(\emptyset) = 0$$

Let now B_k, $k \in \mathbb{N}$ be pairwise disjoint Borel sets. Then $g^{-1}[B_k]$, $k \in \mathbb{N}$, are pairwise disjoint Borel sets and

$$g^{-1}\Big[\bigcup_{k\in\mathbb{N}} B_k\Big] = \bigcup_{k\in\mathbb{N}} g^{-1}[B_k]$$

is a disjoint union (see Tutorial 1). As μ is a measure

$$\mu_g\Big(\bigcup_{k\in\mathbb{N}} B_k\Big) = \mu\Big[g^{-1}\Big[\bigcup_{k\in\mathbb{N}} B_k\Big]\Big) = \mu\Big(\bigcup_{k\in\mathbb{N}} g^{-1}[B_k]\Big) = \sum_{k=0}^{\infty} \mu\big(g^{-1}[B_k]\big) = \sum_{k=0}^{\infty} \mu_g(B_k)$$

Hence μ_g is a Borel measure.

(c) If $f \colon \mathbb{R} \to [0, \infty)$ is simple, prove that

$$\int_X f \circ g \, d\mu = \int_{\mathbb{R}} f \, d\mu_g \tag{1}$$

Solution: Suppose that f is a Borel measurable non-negative simple function attaining the distinct values $a_1, \dots, a_n \geq 0$. Then $A_k := f^{-1}[A_k]$ are Borel measurable and

$$f = \sum_{k=1}^{n} a_k 1_{A_k}.$$

Now $(1_{A_k} \circ g)(x) = 1_{A_k}(g(x)) = 1$ if and only if $x \in g^{-1}[A_k]$, and zero otherwise. Hence

$$f \circ g = \sum_{k=1}^{n} 1_{A_k} \circ g = \sum_{k=1}^{n} 1_{g^{-1}[A_k]}$$

is a simple function and so

$$\int_X f \circ g \, d\mu = \sum_{k=1}^{n} a_k \mu\big(g^{-1}[A_k]\big) = \sum_{k=1}^{n} a_k \mu_g(A_k) = \int_{\mathbb{R}} f \, d\mu_g$$

as claimed.

(d) Prove (1) for every non-negative Borel measurable function $f \colon \mathbb{R} \to [0, \infty]$.

Solution: By a result from lectures there exist simple Borel measurable functions $f_n \colon \mathbb{R} \to \mathbb{R}$ so that $0 \leq f_n \leq f_{n+1} \leq f$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise. Then clearly also $f_n \circ g \colon X \to \mathbb{R}$ so that $0 \leq f_n \leq f_{n+1} \leq f$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise. Hence from the previous part and the monotone convergence theorem

$$\int_X f \circ g \, d\mu = \lim_{n\to\infty} \int_X f_n \circ g \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu_g = \int_{\mathbb{R}} f \, d\mu_g,$$

so (1) follows.

(e) Prove (1) for every Borel measurable function $f: \mathbb{R} \to \mathbb{R}$.

Solution: Observe that $(f \circ g)(x) = f(g(x)) \geq 0$ if and only if $f(g(x)) = f^+(g(x))$, so $f^+ \circ g = (f \circ g)^+$. Similarly with the negative part $f^- \circ g = (f \circ g)^-$. Hence by definition

$$\begin{aligned}\int_X f \circ g \, d\mu &= \int_X (f \circ g)^+ \, d\mu - \int_X (f \circ g)^- \, d\mu \\ &= \int_X f^+ \circ g \, d\mu - \int_X f^- \circ g \, d\mu = \int_{\mathbb{R}} f^+ \, d\mu_g - \int_{\mathbb{R}} f^- \, d\mu_g = \int_{\mathbb{R}} f \, d\mu_g.\end{aligned}$$

(f) If $g: \mathbb{R} \to \mathbb{R}$ is continuously differentiable and μ the Lebesgue measure, identify the measure μ_g. Which method of integration does (1) correspond to in this case?

Solution: This corresponds to integration by substitution

$$\int_{\mathbb{R}} f(g(x)) \, dx = \int_{\mathbb{R}} \frac{f(x)}{g'(x)} \, dx.$$

Extra questions for further practice

4. (a) Let $F(t) := 0$ for $t < 0$ and $F(t) = 1$ for $t \geq 0$. Determine the corresponding Lebesgue-Stieltjes measure.

Solution: Let μ_F by the Lebesgue-Stieltjes measure induced by F. Since intervals of the form $(a, b]$ are measurable and $\mu_F((a, b]) = F(b) - F(a)$ the continuity properties of measures imply

$$\mu_F(\mathbb{R}) = \lim_{n\to\infty} \mu_F((-n, n]) = 1.$$

Similarly,

$$\mu_F((0, \infty)) = \lim_{n\to\infty} \mu_F((0, n]) = \lim_{n\to\infty} (F(n) - F(0)) = 0$$

and

$$\mu_F((-\infty, 0)) = \lim_{n\to\infty} \mu_F((-n, -1/n)) = \lim_{n\to\infty} (F(n) - F(1/n)) = 0.$$

As a consequence, $\mu_F(\{1\}) = 1$. Let now $A \subseteq \mathbb{R}$. If $0 \notin A$, then

$$0 \leq \mu_F^*(A) \leq \mu_F^*(A \cap (-\infty, 0)) + \mu_F^*(A \cap (0, \infty)) \leq \mu_F^*((-\infty, 0)) + \mu_F^*((0, \infty)) = 0.$$

If $1 \in A$, then

$$1 = \mu_f^*(\{1\}) \leq \mu_F^*(A) \leq \mu_F^*(\mathbb{R}) = 1.$$

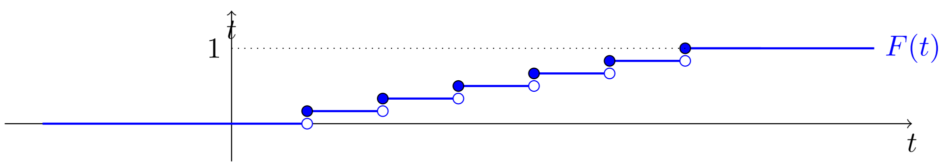
Hence $\mu_F = \delta_0$ is the Dirac measure concentrated at zero.

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ be the sample space for rolling a dice and $P[E] := \mathrm{card}(E)/6$ the probability of the event $E \subseteq \Omega$.

(i) What is the corresponding distribution function $F: \mathbb{R} \to [0, 1]$

Solution: The distribution function is

$$F(t) = \begin{cases} 0 & \text{if } t < 1 \\ k/6 & \text{if } k \leq t < k+1,\ k = 1, \dots, 5 \\ 1 & \text{if } k \geq 6 \end{cases}$$



(ii) What is the measure on $\mathbb{R}$ corresponding to that distribution function.

Solution: By a similar method as in part (a) we get

$$\mu_F = \sum_{k=1}^{6} \frac{k}{6}\delta_k,$$

where δ_k is the Dirac measure concentrated at k.

5. Suppose that $(a, b) \subseteq \mathbb{R}$ is an interval (finite or infinite) and that $f\colon (a, b) \to [0, \infty]$ is measurable.

(a) Show that f is integrable over (a, b) if and only if

$$\lim_{c \to b-} \int_a^c f(x)\, dx < \infty.$$

Further show that in this case

$$\int_a^b f(x)\, dx = \lim_{c \to b-} \int_a^c f(x)\, dx.$$

A similar statement holds if we look at $\lim_{c \to a+} \int_c^b f(x)\, dx$.

Hint: Use the monotone convergence theorem.

(In case of the Riemann integral, the latter is the *definition* of improper integrals. For the Lebesgue integral this provides a way to verify functions are integrable and compute the integrals)

Solution: For every increasing sequence $c_n \to b$ we define $f_n := 1_{(a,c_n]}$. Then $f_{n+1} \geq f_n$ for all $n \in \mathbb{N}$ and $f_n(x) \to f(x)$ for all $x \in (a, b)$. Hence by the monotone convergence theorem

$$\int_a^b f dx = \lim_{n \to \infty} \int_a^b f_n\, dx = \lim_{n \to \infty} \int_a^{c_n} f\, dx.$$

Since this is true for every such sequence (c_n) we conclude that f is integrable if and only if $\lim_{c \to b-} \int_a^c f(x)\, dx < \infty$. Moreover $\int_a^b f(x)\, dx = \lim_{c \to b-} \int_a^c f(x)\, dx$ in that case.

(b) Determine for which $\alpha \in \mathbb{R}$ the following integrals are finite.

(i) $\displaystyle\int_1^\infty \frac{1}{x^\alpha}\, dx$

Solution: By using part (a) we have

$$\int_1^\infty \frac{1}{x^\alpha}\, dx = \lim_{c \to \infty} \int_1^c \frac{1}{x^\alpha}\, dx = \lim_{c \to \infty} \frac{1}{1-\alpha} c^{1-\alpha} < \infty$$

if and only if $\alpha > 1$. If $\alpha = 1$, then

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{c \to \infty} \int_1^c \frac{1}{x}\, dx = \lim_{c \to \infty} \log c = \infty$$

(ii) $\displaystyle\int_0^1 \frac{1}{x^\alpha}\, dx$

Solution: By using part (a) we have

$$\int_0^1 \frac{1}{x^\alpha}\, dx = \lim_{c \to 0+} \int_c^1 \frac{1}{x^\alpha}\, dx = \lim_{c \to 0+} \frac{1}{1-\alpha}(c^{1-\alpha} - 1) < \infty$$

if and only if $\alpha < 1$. If $\alpha = 1$, then

$$\int_0^1 \frac{1}{x}\, dx = \lim_{c \to 0+} \int_c^1 \frac{1}{x}\, dx = \lim_{c \to 0+} (-\log c) = \infty$$

(iii) $\displaystyle\int_0^\infty \frac{1}{x^\alpha}\,dx$

Solution: By the previous part the above integral never exists.

(iv) $\displaystyle\int_0^\infty e^{-\alpha x}\,dx$

Solution:

By using part (a) we have

$$\int_0^\infty e^{-\alpha x}\,dx = \lim_{c\to\infty}\int_1^c e^{-\alpha x}\,dx = \lim_{c\to\infty}\frac{1}{\alpha}(1-e^{-\alpha x}) < \infty$$

if and only if $\alpha > 0$.

6. Let $\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}\,dx$ be the Gamma function defined for $t > 0$.

(a) Prove that $\Gamma(t)$ is differentiable for all $t > 0$.

Solution: If $t > 0$, then

$$\frac{\partial}{\partial t}x^{t-1}e^{-x} = \frac{\partial}{\partial t}e^{(t-1)\log x}e^{-x} = \log x e^{(t-1)\log x}e^{-x} = x^{t-1}e^{-x}\log x.$$

In order to show that Γ is differentiable for $t > 0$ it is sufficient to show that Γ is differentiable on every compact interval $[a,b]$. We now bound the above partial derivative with an integrable function of x only. First assume that $t \in [a,b]$ and $x \geq 1$. Since $0 \leq \log x \leq x$ we get

$$\Big|\frac{\partial}{\partial t}x^{t-1}e^{-x}\Big| \leq x^t e^{-x} \leq x^b e^{-x/2}e^{-x/2}$$

for all $x \geq 1$ and $t \in [a,b]$. Since $x^b e^{-x/2} \to 0$ as $x \to \infty$ there exists a constant $C_b > 0$ such that

$$\Big|\frac{\partial}{\partial t}x^{t-1}e^{-x}\Big| \leq C_b e^{-x/2}$$

for all $x \geq 1$ and $t \in [a,b]$. Now look at $x \in (0,1]$ and $t \in [a,b]$. Since $x^{a/2}\log x \to 0$ there exists a constant $C_a > 0$ such that

$$\Big|\frac{\partial}{\partial t}x^{t-1}e^{-x}\Big| \leq C_a x^{t-1-a/2}e^{-x} \leq C_a x^{a-1-a/2} \leq C_a x^{a/2-1}.$$

Now set $g(x) := C_a x^{a/2-1}$ for $x \in (0,1]$ and $g(x) := C_b e^{-x/2}$ if $x > 1$. Then $g \in \mathcal{L}^1((0,\infty),\mathbb{R})$ and

$$\Big|\frac{\partial}{\partial t}x^{t-1}e^{-x}\Big| \leq g(x)$$

for all $x > 0$ and all $t \in [a,b]$. Hence the theorem on the differentiation of parameter integrals applies, showing that $\Gamma\colon (0,\infty) \to \mathbb{R}$ is differentiable.

(b) Use the formula $\Gamma(t+1) = t\Gamma(t)$ to show that

$$\Gamma(t) = \frac{\Gamma(t+n)}{t(t+1)\cdots(t+n-1)}. \tag{2}$$

for all $n \in \mathbb{N}$.

Solution: We give a proof by induction. For $n = 1$ this follows since $\Gamma(t+1) = t\Gamma(t)$ and so $\Gamma(t) = \Gamma(t+1)/t$. Hence assume that the formula holds for some $n \geq 1$. Then

$$\Gamma(t+n+1) = (t+n)\Gamma(t+n)$$

replacing t by $t+n$ in the formula $\Gamma(t+1) = t\Gamma(t)$. Hence $\Gamma(t+n) = \Gamma(t+n+1)/(t+n)$ and so by the induction hypothesis

$$\Gamma(t) = \frac{\Gamma(t+n)}{t(t+1)\cdots(t+n-1)} = \frac{\Gamma(t+n+1)}{t(t+1)\cdots(t+n-1)(t+(n+1)-1)}$$

as required.

(c) Define $\Gamma(t)$ by (2) for $-n < t < -n+1$, so that Γ is a function on $\mathbb{R} \setminus \{0, -1, -2, \dots\}$. Show that $\Gamma(t+1) = t\Gamma(t)$ on that domain.

Solution: If $t \in (-n, -n+1)$, then $t+1 \in (-(n-1), -(n-1)+1)$. Hence from (2)

$$\Gamma(t+1) = \frac{t\Gamma(t+n)}{(t+1)((t+1)+1)\cdots((t+1)+(n-1)-1)}$$
$$= \frac{\Gamma(t+1+n-1)}{t(t+1)(t+2)\cdots((t+n-1)} = t\Gamma(t).$$

(d) Show that Γ is differentiable on $\mathbb{R} \setminus \{0, -1, -2, \dots\}$.

Solution: We only have to show that Γ is differentiable on $(-n, -n+1)$ for $n \in \mathbb{N}$. However, this is obvious from (2) since $t+n > 0$ and we have shown in (a) that Γ is differentiable on $(0, \infty)$.

7. Let μ be a measure defined on the σ-algebra $\mathcal{A}$ of subsets of X and let $f\colon X \to [0, \infty)$ be a measurable function. For $t > 0$ define $U_t := \{x \in X \colon f(x) > t\}$. The purpose of this question is to prove that

$$\int_X f\, d\mu = \int_0^\infty \mu(U_t)\, dt. \tag{3}$$

This is area/volume by horizontal slicing, often referred to as *Cavalieri's principle*. A special case is the disc method for computing volumes of revolution, where $\mu(U_t)$ corresponds to the area of the disc at level t.

(a) Let $0 < t_1 < t_2 < \cdots < t_n$ and $f = \sum_{k=0}^n t_k 1_{A_k}$ be a measurable simple function. Show that (3) holds.

Solution: If f is simple, then f can be written as

$$\sum_{k=0}^n t_k 1_{A_k},$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n$ and $X = \bigcup_{k=0}^\infty A_k$ is a disjoint union ($A_0 = \emptyset$ is possible). In that case $\mu(U_t) = \mu(U_{t_{k-1}})$ for all $t \in (t_{k-1}, t_k]$, so $t \to \mu(U_t)$ is a simple function as well and we can write

$$\mu(U_t) = \sum_{k=1}^n \mu(U_{t_{k-1}}) 1_{(t_{k-1} - t_k]}. \tag{4}$$

For the same reason $A_k = U_{t_{k-1}} \setminus U_{t_k}$, so $\mu(A_k) = \mu(U_{t_{k-1}}) - \mu(U_{t_k})$. By definition of the integral

$$\int_X f\, d\mu = \sum_{k=0}^n t_k \mu(A_k) = \sum_{k=0}^n t_k \bigl(\mu(U_{t_{k-1}}) - \mu(U_{t_k})\bigr)$$
$$= \sum_{k=0}^n t_k \mu(U_{t_{k-1}}) - \sum_{k=1}^n t_k \mu(U_{t_k}) = \sum_{k=0}^n t_k \mu(U_{t_{k-1}}) - \sum_{k=1}^{n+1} t_{k-1} \mu(U_{t_{k-1}}).$$

Since $t_0 = 0$ and $\mu(U_{t_n}) = 0$ we get

$$\int_X f\, d\mu = \sum_{k=1}^n (t_k - t_{k-1}) \mu(U_{t_{k-1}}) = \sum_{k=1}^n \mu(U_{t_{k-1}}) m((t_{k-1}, t_k]) = \int_0^\infty \mu(t)\, dt$$

by using (4) and the definition of the integral of a simple function.

(b) Prove (3) for every measurable function $f\colon X \to [0,\infty)$.

Solution: As in lectures we construct a sequence of simple functions approximating f from below. For $n \in \mathbb{N}$ and $k = 0, \dots, n2^n - 1$ we set

$$A_{n,k} := \{x \in X\colon k/2^n < f(x) \leq (k+1)/2^n\} \quad \text{and} \quad A_{n,n2^n} := \{x \in X\colon f(x) > n\}.$$

Then define

$$f_n := \sum_{k=0}^{n2^n} \frac{k}{2^n} 1_{A_{n,k}}.$$

We know that $f_n \leq f$ for all $n \in \mathbb{N}$ and that $f_n \to f$ pointwise. For $n \in \mathbb{N}$ and $t \geq 0$ we then set

$$U_{n,t} := \{x \in X\colon f_n(x) > t\}.$$

Fix now $t \geq 0$. Then for ech $n \in \mathbb{N}$ we can choose $k_n \in \mathbb{N}$ such that

$$\frac{k_n - 1}{2^n} < t \leq \frac{k_n}{2^n}.$$

By definition of $U_{n,t}$ we have

$$\mu(U_{n,t}) = \mu(U_{\frac{k_n}{2^n}}) \leq \mu(U_t) \leq \mu(U_{\frac{k_n-1}{2^n}})$$

for all $n \in \mathbb{N}$. Clearly $k_n/2^n$ is increasing and $(k_n - 1)/2^n$ is decreasing to t. Hence by the continuity of measures

$$\lim_{n\to\infty} \mu(U_{n,t}) = \mu\Big(\bigcap_{n\in\mathbb{N}} U_{\frac{k_n}{2^n}}\Big) \leq \mu(U_t) = \mu\Big(\bigcup_{n\in\mathbb{N}} U_{\frac{k_n+1}{2^n}}\Big)$$

Because $t \to \mu(U_t)$ is increasing it is continuous except for at most countably many t. Hence $\mu(U_{n,t}) \to \mu(U_t)$ almost everywhere. Using (a) we have

$$\int_X f_n \, d\mu = \int_0^\infty \mu(U_{n,t}) \, dt$$

and hence (3) follows by applying the monotone convergence theorem on both sides of the above identity.

Note: One can show that the right hand side of (3) is in fact always a (possibly improper) Riemann integral, by giving a similar argument using upper and lower Riemann sums.