THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

# Tutorial 6 (Week 7)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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## Material covered

(1) Continuity and differentiability of parameter integrals

(2) Properties of $L_p$-spaces

(3) Inequalities involving $L_p$-norms

(4) Applications of measure theory to essential supremum

## Outcomes

After completing this tutorial you should

(1) be able to apply the theorems on continuity and differentiability of parameter integrals

(2) be able to work with $L_p$-norms

(3) get an preliminary understanding of $L^\infty$-norms

## Questions to complete during the tutorial

**1.** Suppose that $p, p_1, p_2 \in (1, \infty)$ are such that $1/p = 1/p_1 + 1/p_2$. Use Hölder's inequality to show that for measurable functions $f_1, f_2$

$$\|f_1 f_2\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

**2.** Show that for $|x| \leq 1$

$$\int_0^\infty \frac{\sin t}{e^t - x}\, dt = \sum_{k=1}^\infty \frac{x^{k-1}}{k^2 + 1}.$$

*Hint:* Use a geometric series to expand $\frac{1}{e^t - x}$ in powers of $x$ and the dominated convergence theorem. Calculate the integrals by using that $\sin t = \operatorname{Im} e^{it}$.

**3.** Let $\Gamma(s) := \displaystyle\int_0^\infty x^{s-1} e^{-x}\, dx$ be the Gamma function and $\zeta(s) := \displaystyle\sum_{n=1}^\infty \frac{1}{n^s}$ the Riemann zeta function defined for $s > 1$.

(a) Show that $\displaystyle\frac{1}{n^s}\Gamma(s) = \int_0^\infty x^{s-1} e^{-nx}\, dx$ for all $s > 0$.

(b) Prove that $\displaystyle\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$ for all $s > 1$.

(c) Prove that

$$\zeta'(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} \log x}{e^x - 1}\, dx - \frac{\Gamma'(s)}{\Gamma(s)} \zeta(s)$$

for all $s > 1$.

## Extra questions for further practice

**4.** Let $\mu$ be a measure defined on the $\sigma$-algebra $\mathcal{A}$ of subsets of $X$. Suppose that $\mu(X) < \infty$ and let $1 \leq p < q < \infty$.

(a) Show that $\|f\|_p \leq \mu(X)^{1/p-1/q}\|f\|_q$, that is $L_q(X,\mathbb{C}) \subseteq L_p(X,\mathbb{C})$.

(b) Give an example to show that the above is a proper inclusion if $1 \leq p < q < \infty$. In particular the inequality implies that every function in $L_q(X,\mathbb{C})$ is also in $L_p(X,\mathbb{C})$.

(c) Give an example to show that $L_q(X,\mathbb{C}) \subseteq L_p(X,\mathbb{C})$ is not true in general if $\mu(X) = \infty$.

**5.** Let $\mu$ be a measure on the $\sigma$-algebra $\mathcal{A}$ of subsets of $X$. If $f\colon X \to \mathbb{C}$ is a measurable function we define

$$\|f\|_\infty := \operatorname*{ess-sup}_{x\in\mathbb{X}} |f(x)| := \inf\bigl\{t > 0\colon \mu(\{x \in X\colon |f(x)| > t\}) = 0\bigr\}.$$

We call this the *essential supremum* of $|f|$ and set

$$\mathcal{L}^\infty(X,\mathbb{C}) := \{f\colon X \to \mathbb{C} \mid f \text{ measurable and } \|f\|_\infty < \infty\}.$$

(a) Let $f \in \mathcal{L}^\infty(X,\mathbb{C})$. Show that $N := \{x \in X\colon |f(x)| > \|f\|_\infty\}$ has zero measure, and that $A_\varepsilon := \{x \in X\colon |f(x)| > \|f\|_\infty - \varepsilon\}$ has positive measure for every $\varepsilon > 0$. Hence explain the term "essential supremum".

(b) Let $N$ be the set from (a) and $M$ a set of zero measure with $N \subset M$. Show that $\sup_{x\in X\setminus M} |f(x)| = \text{ess-sup}_{x\in X} |f(x)|$.

(c) For $f, g \in \mathcal{L}^\infty(X,\mathbb{C})$ and $\alpha \in \mathbb{C}$ prove the following:

- (i) Show that $\|f\|_\infty = 0$ if and only if $f(x) = 0$ almost everywhere.
- (ii) $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$
- (iii) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

(d) Prove that Hölder's inequality remains true if $p = 1$ and $q = \infty$, that is,

$$\Bigl|\int_X fg\,d\mu\Bigr| \leq \|f\|_1\|g\|_\infty$$

for all $f \in \mathcal{L}^1(X,\mathbb{C})$ and $g \in \mathcal{L}^\infty(X,\mathbb{C})$.

**6.** Generalise Question **1** as follows: Let $p, p_k \in [1,\infty]$, $k = 1,\dots,n$, with $1/p = 1/p_1 + \cdots + 1/p_n$. Show that for measurable functions $f_1,\dots,f_n$

$$\|f_1 \dots f_n\|_p \leq \|f_1\|_{p_1}\|f_2\|_{p_2}\cdots\|f_n\|_{p_n}.$$

**7.** Suppose that $f_n \to f$ in $L^p(\mathbb{R})$ with $p \in [1,\infty)$ and that $(g_n)$ is a bounded sequence in $L^\infty(\mathbb{R})$ with $g_n \to g$ pointwise. Prove that $f_ng_n \to fg$ in $L^p(\mathbb{R})$.

## Challenge questions (optional)

**8.** Let $\mu$ be a measure on the $\sigma$-algebra $\mathcal{A}$ of subsets of $X$. If $f\colon X \to \mathbb{C}$ and

$$\|f\|_\infty := \operatorname*{ess-sup}_{x\in\mathbb{X}} |f(x)| := \inf\bigl\{\alpha > 0\colon \mu\{x \in X\colon |f(x)| > \alpha\}) = 0\bigr\}.$$

Let $(f_n)$ be a Cauchy sequence in $\mathcal{L}^\infty(X, \mathbb{C})$, that is, with respect to the (essential) supremum norm.

(a) Show that there exists a set $N \subset X$ with $\mu(N) = 0$, so that

$$\sup_{x\in X\setminus N} |f_n(x) - f_m(x)| = \operatorname*{ess-sup}_{x\in X} |f_n(x) - f_m(x)|$$

for all $n, m \in \mathbb{N}$.

(b) If $N$ is the set from the previous part, show that $f_n \to f$ uniformly on $X \setminus N$ for some $f \in \mathcal{L}^\infty(X, \mathbb{C})$.

(c) Let $L^\infty(X, \mathbb{C}) := \{[f]\colon f \in \mathcal{L}^\infty(X, \mathbb{C})\}$, where $[f]$ is the equivalence class of $f$ with respect to the equivalence relation given by $f \sim g$ if $f = g$ almost everywhere. Show that $L^\infty(X, \mathbb{C})$ is a complete normed space.