Material covered

(1) Continuity and differentiability of parameter integrals
(2) Properties of \( L^p \)-spaces
(3) Inequalities involving \( L^p \)-norms
(4) Applications of measure theory to essential supremum

Outcomes

After completing this tutorial you should

(1) be able to apply the theorems on continuity and differentiability of parameter integrals
(2) be able to work with \( L^p \)-norms
(3) get an preliminary understanding of \( L^\infty \)-norms

Summary of essential material

The Lebesgue spaces

Let \((X, \mathcal{A}, \mu)\) be a measure space and \(1 \leq p < \infty\). Define the \( L^p \)-norm by

\[
\|u\|_p := \left( \int_X |u|^p \, d\mu \right)^{1/p}
\]

and

\[
L^p(X) := \{ f : X \to \mathbb{K} \text{ measurable} : \|u\|_p < \infty \}.
\]

Denote by \( \mathcal{N}(X) \) the subspace of functions that are zero almost everywhere. We define

\[
L^p(X) := L^p(X)/\mathcal{N}(X) \quad \text{(vector space quotient)}
\]

This is the space of equivalence classes of functions in \( L^p(X) \) that are equal almost everywhere. Some basic inequalities:

- Hölder’s inequality
  \[
  \int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1
  \]

- Minkowski’s inequality
  \[
  \|f + g\|_p \leq \|f\|_p + \|g\|_p
  \]
Integrals with parameters

Based on the dominated convergence theorem we regularly use the following theorem on the continuity and differentiation of parameter integrals. Important points:

- The most important condition to check is the existence of a dominating functions.
- Continuity and differentiability are local properties. Hence it is sufficient to check on some neighbourhood of every point. For instance to check continuity or differentiability on \((0, \infty)\) it is sufficient to check on every closed interval \([a, b]\), where \(0 < a < b < \infty\).

**Continuity of parameter integrals** Let \((X, \mathcal{A}, \mu)\) a measure space and \(Y\) a metric space (usually a subset of \(\mathbb{R}\) or \(\mathbb{C}\)). Suppose that \(f : X \times Y \to \mathbb{K}\) is a function such that

- \(x \mapsto f(x, y)\) is a measurable function for all \(y \in Y\);
- \(y \mapsto f(x, y)\) is continuous at \(y_0\) for almost all \(x \in X\);
- there exists \(g \in L^1(X, \mathbb{R})\) such that 
  \[ |f(x, y)| \leq g(x) \]
  for almost all \(x \in X\) and all \(y \in Y\). For \(y \in Y\) define
  \[ F(y) := \int_X f(x, y) \, d\mu. \]

Then \(F\) is continuous at \(y_0\).

**Differentiation of parameter integrals** Let \((X, \mathcal{A}, \mu)\) a measure space and \(J \subseteq \mathbb{R}\) an interval. Suppose that \(f : X \times J \to \mathbb{K}\) is a function such that

- \(x \mapsto f(x, t)\) is \(\mu\)-integrable for all \(t \in J\);
- for almost all \(x \in X\), \(\frac{\partial f}{\partial t}(x, t)\) exists and is continuous on \(J\);
- there exists \(g \in L^1(X, \mathbb{R})\) such that 
  \[ \left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x) \]
  for almost all \(x \in X\) and all \(t \in J\). For \(t \in J\) define
  \[ F(t) := \int_X f(x, t) \, d\mu. \]

Then \(F : J \to \mathbb{K}\) is differentiable and

\[ F'(t) := \int_X \frac{\partial f}{\partial t}(x, t) \, d\mu \]
Questions to complete during the tutorial

1. Suppose that \( p, p_1, p_2 \in (1, \infty) \) are such that \( 1/p = 1/p_1 + 1/p_2 \). Use Hölder’s inequality to show that for measurable functions \( f_1, f_2 \)

\[
\|f_1 f_2\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2}.
\]

**Solution:** Set \( q_1 := p_1/p \) and \( q_2 := p_2/p \). Since \( 1/p = 1/p_1 + 1/p_2 \) we have \( p_1, p_2 \geq p \) and so \( q_1, q_2 \geq 1 \). Moreover, \( 1/q_1 + 1/q_2 = p/p_1 + p/p_2 = p/p = 1 \). Hence we can apply Hölder’s inequality:

\[
\|f_1 f_2\|_p^p = \int_X |f_1|^p |f_2|^p d\mu \leq \left( \int_X |f_1|^{pq_1} d\mu \right)^{1/q_1} \left( \int_X |f_2|^{pq_2} d\mu \right)^{1/q_2}.
\]

Since \( pq_i = p_i \) and \( 1/q_i = p/p_i \) we get

\[
\|f_1 f_2\|_p^p = \int_X |f_1|^p |f_2|^p d\mu \leq \left( \int_X |f_1|^{p_1} d\mu \right)^{p/p_1} \left( \int_X |f_2|^{p_2} d\mu \right)^{p/p_2} = \|f_1\|_{p_1}^p \|f_2\|_{p_2}^p.
\]

Hence the assertion follows by taking the \( p \)-th root.

2. Show that for \( |x| \leq 1 \)

\[
\int_0^\infty \frac{\sin t}{e^t - x} \, dt = \sum_{k=1}^\infty \frac{x^{k-1}}{k^2 + 1}.
\]

**Hint:** Use a geometric series to expand \( \frac{1}{e^t - x} \) in powers of \( x \) and the dominated convergence theorem. Calculate the integrals by using that \( \sin t = \text{Im} e^{it} \).

**Solution:** Note that \( |e^{-t} x| < 1 \) whenever \( |x| \leq 1 \) and \( t > 0 \). Hence if we fix \( |x| \leq 1 \), then by the formula for the geometric series

\[
f(t) := \frac{\sin t}{e^t - x} = e^{-t} \sin t \frac{1}{1 - e^{-t} x} = e^{-t} \sin t \sum_{k=0}^\infty (e^{-t} x)^k = \sum_{k=0}^\infty \sin te^{-(k+1)t} x^k.
\]

for all \( t > 0 \). For \( n \in \mathbb{N} \) we define

\[
f_n(x, t) := \sum_{k=0}^n \sin te^{-(k+1)t} x^k
\]

for all \( t > 0 \). Then \( f_n(t) \to f(t) \) for all \( t > 0 \) and

\[
|f_n(x, t)| \leq \sum_{k=0}^n |\sin t|e^{-(k+1)t} |x|^k \leq |\sin t|e^{-t} \sum_{k=0}^\infty e^{-kt} = \frac{|\sin t|}{e^t - 1} \leq g(t) := \frac{t}{e^t - 1}
\]

for all \( n \in \mathbb{N} \) since \( |x| \leq 1 \) and \( |\sin t| \leq t \) for all \( t > 0 \). As \( g(t) \to 1 \) as \( t \to 0 \) and by the exponential series

\[
g(t) = \frac{t}{e^t - 1} = \frac{t}{t + t^2/2 + t^3/3! + \ldots} \leq \frac{6}{t^2}
\]

for all \( t > 0 \), the function \( g \) is integrable over \( (0, \infty) \). Hence by the dominated convergence theorem

\[
\int_0^\infty \frac{\sin t}{e^t - x} \, dt = \lim_{n \to \infty} \int_0^\infty f_n(t) \, dt = \sum_{k=0}^\infty \int_0^\infty \sin te^{-(k+1)t} \, dt x^k.
\]
To compute the integrals we note that \( \sin t = \text{Im} e^{it} \) and so

\[
\int_0^\infty \sin t e^{-(k+1)it} dt = \text{Im} \int_0^\infty e^{it} e^{-(k+1)it} dt = \text{Im} \int_0^\infty e^{-(k+1-it)} dt.
\]

Now clearly

\[
\int_0^\infty e^{-(k+1-it)} dt = -\frac{1}{k+1-i} e^{-(k+1-it)} \bigg|_0^\infty = \frac{1}{k+1-i} = \frac{k+1+i}{(k+1)^2 + 1}
\]

for all \( k \in \mathbb{N} \). Therefore

\[
\int_0^\infty \sin te^{-(k+1)t} dt = \text{Im} \frac{k+1+i}{(k+1)^2 + 1} = \frac{1}{(k+1)^2 + 1}
\]

and so

\[
\int_0^\infty \frac{\sin t}{e^t - x} dt = \sum_{k=0}^\infty \frac{1}{(k+1)^2 + 1} \frac{x^k}{k^2 + 1}
\]

as claimed.

3. Let \( \Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx \) be the Gamma function defined for \( t > 0 \).

(a) Prove that \( \Gamma(t) \) is differentiable for all \( t > 0 \).

Solution: If \( t > 0 \), then

\[
\frac{\partial}{\partial t} x^{t-1} e^{-x} = \frac{\partial}{\partial t} e^{(t-1)\log x} e^{-x} = \log x e^{(t-1)\log x} e^{-x} = x^{t-1} e^{-x} \log x.
\]

In order to show that \( \Gamma \) is differentiable for \( t > 0 \) it is sufficient to show that \( \Gamma \) is differentiable on every compact interval \([a, b]\). We now bound the above partial derivative with an integrable function of \( x \) only. First assume that \( t \in [a, b] \) and \( x \geq 1 \). Since \( 0 \leq \log x \leq x \) we get

\[
\left| \frac{\partial}{\partial t} x^{t-1} e^{-x} \right| \leq x^{t-1} e^{-x} \leq x^a e^{-x/2} e^{-x/2}
\]

for all \( x \geq 1 \) and \( t \in [a, b] \). Since \( x^a e^{-x/2} \to 0 \) as \( x \to \infty \) there exists a constant \( C_a > 0 \) such that

\[
\left| \frac{\partial}{\partial t} x^{t-1} e^{-x} \right| \leq C_a e^{-x/2}
\]

for all \( x \geq 1 \) and \( t \in [a, b] \). Now look at \( x \in (0, 1] \) and \( t \in [a, b] \). Since \( x^{a/2} \log x \to 0 \) as \( x \to 0 \) there exists a constant \( C_a > 0 \) such that

\[
\left| \frac{\partial}{\partial t} x^{t-1} e^{-x} \right| \leq C_a x^{t-1-a/2} e^{-x} \leq C_a x^{a-1-a/2} \leq C_a x^{a/2-1}.
\]

Now set \( g(x) := C_a x^{a/2-1} \) for \( x \in (0, 1] \) and \( g(x) := C_a x^{-x/2} \) if \( x > 1 \). Then \( g \in L^1((0, \infty), \mathbb{R}) \) and

\[
\left| \frac{\partial}{\partial t} x^{t-1} e^{-x} \right| \leq g(x)
\]

for all \( x > 0 \) and all \( t \in [a, b] \). Hence the theorem on the differentation of parameter integrals applies, showing that \( \Gamma : (0, \infty) \to \mathbb{R} \) is differentiable.

(b) Use the formula \( \Gamma(t+1) = t\Gamma(t) \) to show that

\[
\Gamma(t) = \frac{\Gamma(t+n)}{t(t+1)\cdots(t+n-1)}.
\]

(1)
for all \( n \in \mathbb{N} \).

**Solution:** We give a proof by induction. For \( n = 1 \) this follows since \( \Gamma(t + 1) = t\Gamma(t) \) and so \( \Gamma(t) = \Gamma(t + 1)/t \). Hence assume that the formula holds for some \( n \geq 1 \). Then

\[
\Gamma(t + n + 1) = (t + n)\Gamma(t + n)
\]

Replacing \( t \) by \( t + n \) in the formula \( \Gamma(t + 1) = t\Gamma(t) \). Hence \( \Gamma(t + n) = \Gamma(t + n + 1)/(t + n) \) and so by the induction hypothesis

\[
\Gamma(t) = \frac{\Gamma(t + n)}{t(t + 1) \cdots (t + n - 1)} = \frac{\Gamma(t + n + 1)}{t(t + 1) \cdots (t + n - 1)(t + (n + 1) - 1)}
\]

as required.

(c) Define \( \Gamma(t) \) by (1) for \( -n < t < -(n + 1) \), so that \( \Gamma \) is a function on \( \mathbb{R} \setminus \{0, -1, -2, \ldots\} \).

Show that \( \Gamma(t + 1) = t\Gamma(t) \) on that domain.

**Solution:** If \( t \in (-n, -(n + 1)) \), then \( t + 1 \in (-n - 1, -(n + 1)) \). Hence from (1)

\[
\Gamma(t + 1) = \frac{t\Gamma(t + n)}{(t + 1)((t + 1) + 1) \cdots ((t + 1) + (n - 1) - 1)} = \frac{\Gamma(t + 1 + n)}{t(t + 1)(t + 2) \cdots ((t + n - 1) = t\Gamma(t)}.
\]

(d) Show that \( \Gamma \) is differentiable on \( \mathbb{R} \setminus \{0, -1, -2, \ldots\} \).

**Solution:** We only have to show that \( \Gamma \) is differentiable on \( (-n, -(n + 1)) \) for \( n \in \mathbb{N} \). However, this is obvious from (1) since \( t + n > 0 \) and we have shown in (a) that \( \Gamma \) is differentiable on \( (0, \infty) \).

### Extra questions for further practice

4. Let \( \Gamma(s) := \int_0^\infty x^{s-1} e^{-x} \, dx \) be the Gamma function and \( \zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s} \) the Riemann zeta function defined for \( s > 1 \).

(a) Show that \( \frac{1}{n^s}\Gamma(s) = \int_0^\infty x^{s-1} e^{-nx} \, dx \) for all \( s > 0 \).

**Solution:** If we make the substitution \( y = nx \), then \( dx = dy/n \) and so

\[
\frac{1}{n^s}\Gamma(s) = \int_0^\infty \left(\frac{x}{n}\right)^{s-1} e^{-x} \, dx = \int_0^\infty y^{s-1} e^{-ny} \, dy
\]

(b) Prove that \( \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx \) for all \( s > 1 \).

**Solution:** By the previous part and the definition of the \( \zeta \) function we have

\[
\Gamma(s)\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{-nx} \, dx.
\]

Since all terms in the series are non-negative we can apply the monotone convergence theorem and interchange integration and summation to get

\[
\Gamma(s)\zeta(s) = \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{-nx} \, dx = \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx} \, dx.
\]
From the formula for the geometric series
\[
\sum_{n=1}^{\infty} e^{-nx} = \sum_{n=1}^{\infty} (e^{-x})^n = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1}
\]
and the formula follows.

\(*c)* Prove that
\[
\zeta'(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} \log x}{e^x - 1} \, dx - \frac{\Gamma'(s)}{\Gamma(s)} \zeta(s)
\]
for all \(s > 1\).

**Solution:** We use the theorem on the differentiation of parameter integrals and show that
\[
\frac{\partial}{\partial s} \frac{x^{s-1}}{e^x - 1} = \frac{\partial}{\partial s} \frac{e^{(s-1)\log x}}{e^x - 1} = \frac{x^{s-1} \log x}{e^x - 1}
\]
is dominated by an integrable function uniformly with respect to \(s\) in compact intervals \([a, b]\) with \(1 < a < b < \infty\). First look at \(x \geq 1\). Since \(\log x \leq x\) we get from the exponential series
\[
0 \leq \frac{x^{s-1} \log x}{e^x - 1} \leq \frac{x^{b-1} x}{e^x - 1} \leq \frac{x^b}{x + x^2/2 + x^3/3! + \ldots + x^n/n!} \leq \frac{x^n n!}{x^n} = n! x^{b-1-n}
\]
for all \(x \geq 1\) and \(s \in [a, b]\). We choose \(n\) so that \(b-1-n < -2\). Consider now \(x \in (0, 1)\). Let \(0 < \varepsilon < a - 1\). Since \(x^\varepsilon \log x \to 0\) as \(x \to 0\) there exists a constant \(C_1 > 0\) such that \(x^\varepsilon \log x \leq C_1\) for all \(x \in (0, 1)\). Using that \(e^x - 1 \geq e^x - x\) we get
\[
\frac{x^{s-1} | \log x|}{e^x - 1} \leq C_0 C_1 x^{s-2} \leq C_0 C_1 x^{a-2}
\]
for all \(x \in (0, 1)\) and \(s \in [a, b]\). By choice of \(\varepsilon\) we have \(a - \varepsilon - 2 > -1\). so
\[
\int_0^1 x^{a-2} \, dx < \infty.
\]
Therefore
\[
\left| \frac{\partial}{\partial s} \frac{x^{s-1}}{e^x - 1} \right| = \frac{x^{s-1} | \log x|}{e^x - 1} \leq g(x) := \begin{cases} C_0 C_1 x^{a-2} & \text{if } x \in (0, 1) \\ n! x^{b-1-n} & \text{if } x \geq 1 \end{cases}
\]
for all \(s \in [a, b]\). Since \(g \in L^1((0, \infty), \mathbb{R})\) the theorem on the differentiation of parameter integrals applies.

5. Let \(\mu\) be a measure defined on the \(\sigma\)-algebra \(\mathcal{A}\) of subsets of \(X\). Supposes that \(\mu(X) < \infty\) and let \(1 \leq p < q < \infty\).

(a) Show that \(\|f\|_p \leq \mu(X)^{1/p-1/q} \|f\|_q\), that is \(L_q(X, \mathbb{C}) \subseteq L_p(X, \mathbb{C})\).

**Solution:** We use Hölder’s inequality writing \(|f|^p = 1 \cdot |f|^p\). To get the \(L^q\)-norm we choose one exponent to be \(r = \frac{q}{p} > 1\) and compute the other by solving the equation \(1/r + 1/s = 1\). If we do that we get \(s = \frac{q}{q-p}\). Hence by Hölder’s inequality
\[
\|f\|^p_p = \int_X 1 \cdot |f|^p d\mu \leq \left( \int_X 1^{\frac{q}{q-p}} d\mu \right)^{\frac{p}{q-p}} \left( \int_X |f|^{q/p} d\mu \right)^{\frac{p}{q}} = \mu(X)^{1-\frac{p}{q}} \|f\|^q_q.
\]
Taking the above to the \(1/p\)-th power we get the required inequality.
(b) Give an example to show that the above can be a proper inclusion if \( 1 \leq p < q < \infty \). In particular the inequality implies that every function in \( L_q(X, \mathbb{C}) \) is also in \( L_p(X, \mathbb{C}) \).

**Solution:** Let \( 1 \leq p < q \) and choose \( \alpha > 0 \) such that \( 1/q < \alpha < 1/p \). Set \( f(x) := x^{-\alpha} \) for \( x \in (0, 1] \) and zero otherwise. Then

\[
\|f\|_p = \int_0^1 x^{-\alpha p} \, dx < \infty
\]

since \( \alpha p < 1 \) but

\[
\|f\|_q = \int_0^1 x^{-\alpha q} \, dx = \infty
\]

since \( \alpha q > 1 \) (explicit calculation).

(c) Give an example to show that \( L_q(X, \mathbb{C}) \subset L_p(X, \mathbb{C}) \) is not true in general if \( \mu(X) = \infty \).

**Solution:** Let \( 1 \leq p < q \) and choose \( \alpha > 0 \) such that \( 1/q < \alpha < 1/p \). Set \( f(x) := x^{-\alpha} \) for \( x \in [1, \infty] \) and zero otherwise. Then

\[
\|f\|_p = \int_1^\infty x^{-\alpha p} \, dx = \infty
\]

since \( \alpha p < 1 \) but

\[
\|f\|_q = \int_1^\infty x^{-\alpha q} \, dx < \infty
\]

since \( \alpha q > 1 \) (explicit calculation).

6. Let \( \mu \) be a measure on the \( \sigma \)-algebra \( \mathcal{A} \) of subsets of \( X \). If \( f : X \to \mathbb{C} \) is a measurable function we define

\[
\|f\|_\infty := \text{ess sup}_{x \in X} |f(x)| := \inf \{ t > 0 : \mu(\{ x \in X : |f(x)| > t \}) = 0 \}.
\]

We call this the essential supremum of \( |f| \) and set

\[
L^\infty(X, \mathbb{C}) := \{ f : X \to \mathbb{C} \mid f \text{ measurable and } \|f\|_\infty < \infty \}.
\]

(a) Let \( f \in L^\infty(X, \mathbb{C}) \). Show that \( N := \{ x \in X : |f(x)| > \|f\|_\infty \} \) has zero measure, and that \( A_\varepsilon := \{ x \in X : |f(x)| > \|f\|_\infty - \varepsilon \} \) has positive measure for every \( \varepsilon > 0 \). Hence explain the term “essential supremum”.

**Solution:** For every \( n \in \mathbb{N} \) we set

\[
N_n := \{ x \in X : |f(x)| > \|f\|_\infty + 1/n \}
\]

Then by definition of the essential supremum \( \mu(N_n) = 0 \) for all \( n \in \mathbb{N} \). Clearly \( \bigcup_{n \in \mathbb{N}} N_n = N \) and \( N_n \subset N_{n+1} \) for all \( n \in \mathbb{N} \). By the monotonicity properties of measures

\[
\mu(N) = \mu\left( \bigcup_{n \in \mathbb{N}} N_n \right) = \lim_{n \to \infty} \mu(N_n) = 0
\]

as claimed. By definition of \( \|f\|_\infty \) we have \( \mu(A_\varepsilon) > 0 \) for all \( \varepsilon > 0 \). Hence \( f \) is larger than \( \|f\|_\infty \) only on a set of measure zero. Moreover, \( f \) is larger than any number smaller than \( \|f\|_\infty \) on a “substantially larger” set, namely a set of positive measure. In particular, modifying \( f \) on a set of measure zero preserves the essential supremum, so it is the measure theoretic supremum of a function.
(b) Let \( N \) be the set from (a) and \( M \) a set of zero measure with \( N \subset M \). Show that
\[
\sup_{x \in X \setminus M} |f(x)| = \text{ess} \sup_{x \in X} |f(x)|.
\]
Solution: From the definition of \( N \) it is clear that \(|f(x)| \leq \|f\|_\infty\), that is,
\[
\sup_{x \in X \setminus M} |f(x)| \leq \text{ess} \sup_{x \in X} |f(x)| = \|f\|_\infty.
\]
Clearly \( \mu(A_x \cap M^c) = \mu(A_x) > 0 \) for all \( \epsilon > 0 \). Hence, if we fix \( \epsilon > 0 \), then there exists \( x \in X \setminus M \) such that \(|f(x)| > \|f\|_\infty - \epsilon \) and therefore \( \sup_{x \in X \setminus M} |f(x)| \geq \|f\|_\infty - \epsilon \).
Since this is true for all \( \epsilon > 0 \) we get
\[
\sup_{x \in X \setminus M} |f(x)| \geq \text{ess} \sup_{x \in X} |f(x)| = \|f\|_\infty.
\]

(c) For \( f, g \in L^\infty(X, \mathbb{C}) \) and \( a \in \mathbb{C} \) prove the following:
(i) Show that \( \|f\|_\infty = 0 \) if and only if \( f(x) = 0 \) almost everywhere.
Solution: By (a) the set \( \{ x \in X : |f(x)| > \|f\|_\infty = 0 \} \) has zero measure, so \( f = 0 \) almost everywhere.
(ii) \( \|af\|_\infty = |a|\|f\|_\infty \)
Solution: The assertion is obvious for \( a = 0 \), so assume that \( a \neq 0 \). Then \( |af(x)| > t \) if and only if \( |f(x)| > s = t/|a| \). Hence
\[
\|af\|_\infty = \inf \{ t > 0 : \mu(\{ x \in X : |af(x)| > t \}) = 0 \}
= \inf \{ t > 0 : \mu(\{ x \in X : f(x) > t/|a| \}) = 0 \}
= \inf \{ s |a| > 0 : \mu(\{ x \in X : f(x) > s \}) = 0 \}
= |a| \inf \{ s > 0 : \mu(\{ x \in X : |f(x)| > s \}) = 0 \}
= |a|\|f\|_\infty
\]
as claimed.
(iii) \( \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \)
Solution: We set \( N_1 := \{ x \in X : |f(x)| > \|f\|_\infty \} \), \( N_2 := \{ x \in X : |g(x)| > \|f\|_\infty \} \) and \( N_3 := \{ x \in X : |f(x) + g(x)| > \|f + g\|_\infty \} \). Then let \( M := N_1 \cup N_2 \cup N_3 \). Then by (a) and (b) we have \( \mu(M) = 0 \) and
\[
\|f\|_\infty = \sup_{x \in X \setminus M} |f(x)|
\]
\[
\|g\|_\infty = \sup_{x \in X \setminus M} |g(x)|
\]
\[
\|f + g\|_\infty = \sup_{x \in X \setminus M} |f(x) + g(x)|.
\]
Hence
\[
|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty
\]
for all \( x \in X \setminus M \). Taking the supremum on the left hand side we get
\[
\|f + g\|_\infty = \sup_{x \in X \setminus M} |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty
\]
as claimed.

(d) Prove that Hölder’s inequality remains true if \( p = 1 \) and \( q = \infty \), that is,
\[
\left| \int_X fg \, d\mu \right| \leq \|f\|_1 \|g\|_\infty
\]
The aim of the question below is to guide you through the proof of the completeness of the space $L^\infty(X, \mathbb{C})$.

**Solution:** By (b) we have

$$\left| \int_X fg\ d\mu \right| \leq \int_X |f||g|\ d\mu = \int_{X\setminus N} |f||g|\ d\mu \leq \int_{X\setminus N} |g| \sup_{x\in X\setminus N} |f(x)|\ d\mu = \|f\|_1\|g\|_\infty$$

7. Generalise Question 1 as follows: Let $p, p_k \in [1, \infty]$, $k = 1, \ldots, n$, with $1/p = 1/p_1 + \cdots + 1/p_n$. Show that for measurable functions $f_1, \ldots, f_n$

$$\|f_1 \cdots f_n\|_p \leq \|f_1\|_{p_1}\|f_2\|_{p_2} \cdots \|f_n\|_{p_n}.$$  

**Solution:** We use a proof by induction. If $n = 2$ the inequality is proved in Question 1. Assume the assertion holds for $n - 1$ functions. Define $q$ by

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}}.$$  

Then clearly $1/p = 1/q + 1/p_n$ and so we can apply Question 1 and the induction hypothesis to get

$$\|f_1 \cdots f_n\|_p \leq \|f_1 \cdots f_{n-1}\|_q\|f_n\|_{p_n} \leq (\|f_1\|_{p_1}\|f_2\|_{p_2} \cdots \|f_{n-1}\|_{p_{n-1}})\|f_n\|_{p_n}$$

as required.

8. Suppose that $f_n \to f$ in $L^p(\mathbb{R})$ with $p \in [1, \infty)$ and that $(g_n)$ is a bounded sequence in $L^\infty(\mathbb{R})$ with $g_n \to g$ pointwise. Prove that $f_ng_n \to fg$ in $L^p(\mathbb{R})$.

**Solution:** By assumption there exists $M > 0$ such that $\|g_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Hence we have

$$\|f_ng_n - fg\|_p \leq \|(f_n - f)g_n\|_p + \|f(g_n - g)\|_p$$

for all $n \in \mathbb{N}$. First note that

$$\|(f_n - f)g_n\|_p \leq \|g_n\|_\infty\|f_n - f\|_p \leq M\|f_n - f\|_p \to 0$$

as $n \to \infty$. Second note that $|f(g_n - g)|^p \leq (2M)^p|f|^p$ almost everywhere. Moreover, $f g_n \to fg$ pointwise and $(2M)^p|f|^p \in L^1(\mathbb{R})$ and so by the dominated convergence theorem

$$\|f(g_n - g)\|_p^p = \int_{-\infty}^{\infty} |f(g_n - g)|^p\ dx \to 0.$$

**Challenge questions (optional)**

The aim of the question below is to guide you through the proof of the completeness of the space $L^\infty(X)$ with respect to the essential supremum norm.

9. Let $\mu$ be a measure on the $\sigma$-algebra $\mathcal{A}$ of subsets of $X$. If $f : X \to \mathbb{C}$ and

$$\|f\|_\infty := \text{ess sup}_{x \in X} |f(x)| := \inf \left\{ \alpha > 0 : \mu \{ x \in X : |f(x)| > \alpha \} = 0 \right\}.$$  

Let $(f_n)$ be a Cauchy sequence in $L^\infty(X, \mathbb{C})$, that is, with respect to the (essential) supremum norm.
(a) Show that there exists a set $N \subset X$ with $\mu(N) = 0$, so that

$$\sup_{x \in X \setminus N} |f_n(x) - f_m(x)| = \operatorname{ess sup}_{x \in X} |f_n(x) - f_m(x)|$$

for all $n, m \in \mathbb{N}$.

**Solution:** For $n, m \in \mathbb{N}$ define the sets $A_{n,m} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$, where $\|f_n - f_m\|_\infty$ is the essential supremum norm. By Question 6(a) we have $\mu(A_{n,m}) = 0$ and if we set $N := \bigcup_{n,m \in \mathbb{N}} A_{n,m}$ then also $\mu(N) = 0$. Now Question 6(b) implies that

$$\operatorname{ess sup}_{x \in X} |f_n(x) - f_m(x)| = \sup_{x \in X \setminus N} |f_n(x) - f_m(x)|$$

for all $n, m \in \mathbb{N}$ for the above set $N$ defined above.

(b) If $N$ is the set from the previous part, show that $f_n \to f$ uniformly on $X \setminus N$ for some $f \in L^\infty(X, \mathbb{C})$.

**Solution:** The sequence $(f_n)$ is uniformly Cauchy on $X \setminus N$ and therefore by a result from analysis it converges uniformly to some bounded function $f$ defined on $X \setminus N$. We define $f(x) := 0$ for $x \in N$. Then clearly $1_{X \setminus N} f_n \to f$ pointwise on $X$ and since $1_{X \setminus N} f_n$ is measurable the pointwise limit $f$ is measurable as well. Hence $f \in L^\infty(X, \mathbb{C})$.

(c) Let $L^\infty(X, \mathbb{C}) := \left\{ [f] : f \in L^\infty(X, \mathbb{C}) \right\}$, where $[f]$ is the equivalence class of $f$ with respect to the equivalence relation given by $f \sim g$ if $f = g$ almost everywhere. Show that $L^\infty(X, \mathbb{C})$ is a complete normed space.

**Solution:** By Question 6(c) $L^\infty(X, \mathbb{C})$ is a normed space. Let $[f_n]$ be a Cauchy sequence with $f_n \in L^\infty(X, \mathbb{C})$. By the previous part there exists $f \in L^\infty(X, \mathbb{C})$ with $\|f_n - f\|_\infty \to 0$. A similar assertion holds if $\tilde{f}_n \in [f_n]$ are different representatives of the equivalence class. Then there exists $\tilde{f}$ with $\|\tilde{f}_n - \tilde{f}\|_\infty \to 0$. Since $f_n = \tilde{f}_n$ almost everywhere we get

$$\|f - \tilde{f}\|_\infty \leq \|f - f_n\|_\infty + \|f_n - \tilde{f}_n\|_\infty = \|f - f_n\|_\infty + \|\tilde{f}_n - \tilde{f}\|_\infty \to 0$$

as $n \to \infty$. Hence $\|f - \tilde{f}\|_\infty = 0$ and therefore $f = \tilde{f}$ almost everywhere. Hence $f_n \to f$ (or more precisely $[f_n] \to [f]$ in $L^\infty(X, \mathbb{C})$).