The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 6 (Week 7)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) Continuity and differentiability of parameter integrals

(2) Properties of L_p-spaces

(3) Inequalities involving L_p-norms

(4) Applications of measure theory to essential supremum

Outcomes

After completing this tutorial you should

(1) be able to apply the theorems on continuity and differentiability of parameter integrals

(2) be able to work with L_p-norms

(3) get an preliminary understanding of L^∞-norms

Questions to complete during the tutorial

1. Suppose that $p, p_1, p_2 \in (1, \infty)$ are such that $1/p = 1/p_1 + 1/p_2$. Use Hölder's inequality to show that for measurable functions f_1, f_2

$$\|f_1 f_2\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Solution: Set $q_1 := p_1/p$ and $q_2 := p_2/p$. Since $1/p = 1/p_1 + 1/p_2$ we have $p_1, p_2 \geq p$ and so $q_1, q_2 \geq 1$. Moreover, $1/q_1 + 1/q_2 = p/p_1 + p/p_2 = p/p = 1$. Hence we can apply Hölder's inequality:

$$\|f_1 f_2\|_p^p = \int_X |f_1|^p |f_2|^p \, d\mu \leq \Big(\int_X |f_1|^{pq_1} \, d\mu\Big)^{1/q_1} \Big(\int_X |f_2|^{pq_2} \, d\mu\Big)^{1/q_1}.$$

Since $pq_i = p_i$ and $1/q_i = p/p_i$ we get

$$\|f_1 f_2\|_p^p = \int_X |f_1|^p |f_2|^p \, d\mu \leq \Big(\int_X |f_1|^{p_1} \, d\mu\Big)^{p/p_1} \Big(\int_X |f_2|^{p_2} \, d\mu\Big)^{p/p_1} = \|f_1\|_{p_1}^p \|f_2\|_{p_2}^p.$$

Hence the assertion follows by taking the p-th root.

2. Show that for $|x| \leq 1$

$$\int_0^\infty \frac{\sin t}{e^t - x} \, dt = \sum_{k=1}^\infty \frac{x^{k-1}}{k^2 + 1}.$$

Hint: Use a geometric series to expand $\frac{1}{e^t - x}$ in powers of x and the dominated convergence theorem. Calculate the integrals by using that $\sin t = \operatorname{Im} e^{it}$.

Solution: Note that $|e^{-t}x| < 1$ whenever $|x| \leq 1$ and $t > 0$. Hence if we fix $|x| \leq 1$, then by the formula for the geometric series

$$f(t) := \frac{\sin t}{e^t - x} = e^{-t} \sin t \frac{1}{1 - e^{-t}x} = e^{-t} \sin t \sum_{k=0}^{\infty} (e^{-t}x)^k = \sum_{k=0}^{\infty} \sin t e^{-(k+1)t} x^k.$$

for all $t > 0$. For $n \in \mathbb{N}$ we define

$$f_n(t) := \sum_{k=0}^{n} \sin t e^{-(k+1)t} x^k$$

for all $t > 0$. Then $f_n(t) \to f(t)$ for all $t > 0$ and

$$|f_n(x)| \leq \sum_{k=0}^{n} |\sin t| e^{-(k+1)t} |x|^k \leq |\sin t| e^{-t} \sum_{k=0}^{\infty} e^{-kt} = \frac{|\sin t|}{e^t - 1} \leq g(t) := \frac{t}{e^t - 1}$$

for all $n \in \mathbb{N}$ since $|x| \leq 1$ and $|\sin t| \leq t$ for all $t > 0$. As $g(t) \to 1$ as $t \to 0$ and by the exponential series

$$g(t) = \frac{t}{e^t - 1} = \frac{t}{t + t^2/2 + t^3/3! + \dots} \leq \frac{6}{t^2}$$

for all $t > 0$, the function g is integrable over $(0, \infty)$. Hence by the dominated convergence theorem

$$\int_0^{\infty} \frac{\sin t}{e^t - x} \, dt = \lim_{n \to \infty} \int_0^{\infty} f_n(t) \, dt = \sum_{k=0}^{\infty} \int_0^{\infty} \sin t e^{-(k+1)t} \, dt \; x^k.$$

To compute the integrals we note that $\sin t = \operatorname{Im} e^{it}$ and so

$$\int_0^{\infty} \sin t e^{-(k+1)t} \, dt = \operatorname{Im} \int_0^{\infty} e^{it} e^{-(k+1)t} \, dt = \operatorname{Im} \int_0^{\infty} e^{-(k+1-i)t} \, dt.$$

Now clearly

$$\int_0^{\infty} e^{-(k+1-i)t} \, dt = -\frac{1}{k+1-i} e^{-(k+1-i)t} \Big|_0^{\infty} = \frac{1}{k+1-i} = \frac{k+1+i}{(k+1)^2 + 1}$$

for all $k \in \mathbb{N}$. Therefore

$$\int_0^{\infty} \sin t e^{-(k+1)t} \, dt = \operatorname{Im} \frac{k+1+i}{(k+1)^2 + 1} = \frac{1}{(k+1)^2 + 1}$$

and so

$$\int_0^{\infty} \frac{\sin t}{e^t - x} \, dt = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 + 1} x^k = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k^2 + 1}$$

as claimed.

3. Let $\Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x} \, dx$ be the Gamma function and $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ the Riemann zeta function defined for $s > 1$.

(a) Show that $\frac{1}{n^s} \Gamma(s) = \int_0^{\infty} x^{s-1} e^{-nx} \, dx$ for all $s > 0$.

Solution: If we make the substitution $y = nx$, then $dx = dy/n$ and so

$$\frac{1}{n^s} \Gamma(s) = \int_0^{\infty} \left(\frac{x}{n}\right)^{s-1} e^{-x} n \, dx = \int_0^{\infty} y^{s-1} e^{-ny} \, dy$$

(b) Prove that $\zeta(s) = \dfrac{1}{\Gamma(s)} \displaystyle\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx$ for all $s > 1$.

Solution: By the previous part and the definition of the ζ function we have

$$\Gamma(s)\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}\Gamma(s) = \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\, dx.$$

Since all terms in the series are non-negative we can apply the monotone convergence theorem and interchange integration and summation to get

$$\Gamma(s)\zeta(s) = \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\, dx = \int_0^\infty x^{s-1} \sum_{n=1}^\infty e^{-nx}\, dx.$$

From the formula for the geometric series

$$\sum_{n=1}^\infty e^{-nx} = \sum_{n=1}^\infty (e^{-x})^n = \frac{e^{-1}}{1 - e^{-x}} = \frac{1}{e^x - 1}$$

and the formula follows.

(c) Prove that

$$\zeta'(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\log x}{e^x - 1}\, dx - \frac{\Gamma'(s)}{\Gamma(s)}\zeta(s)$$

for all $s > 1$.

Solution: We use the theorem on the differentiation of parameter integrals and show that

$$\frac{\partial}{\partial s}\frac{x^{s-1}}{e^x - 1} = \frac{\partial}{\partial s}\frac{e^{(s-1)\log x}}{e^x - 1} = \frac{x^{s-1}\log x}{e^x - 1}$$

is dominated by an integrable function uniformly with respect to s in compact intervals $[a, b]$ with $1 < a < b < \infty$. First look at $x \geq 1$. Since $\log x \leq x$ we get from the exponential series

$$0 \leq \frac{x^{s-1}\log x}{e^x - 1} \leq \frac{x^{b-1}x}{e^x - 1} \leq \frac{x^b}{x + x^2/ + x^3/3! + ... + x^n/n!} \leq \frac{x^b n!}{x^n} = n!x^{b-1-n}$$

for all $x \geq 1$ and $s \in [a, b]$. We choose n so that $b - 1 - n < -2$. Consider now $x \in (0, 1)$. Let $0 < \varepsilon < a - 1$. Since $x^\varepsilon \log x \to 0$ as $x \to 0$ there exists a constant $C_1 > 0$ such that $x^\varepsilon |\log x| \leq C_1$ for all $x \in (0, 1)$. Using that $e^x - 1 \geq x$ we get

$$\frac{x^{s-1}|\log x|}{e^x - 1} \leq C_0C_1x^{s-\varepsilon-2} \leq C_0C_1x^{a-\varepsilon-2}$$

for all $x \in (0, 1)$ and $s \in [a, b]$. By choice of ε we have $a - \varepsilon - 2 > -1$. so

$$\int_0^1 x^{a-\varepsilon-2}\, dx < \infty.$$

Therefore

$$\Big|\frac{\partial}{\partial s}\frac{x^{s-1}}{e^x - 1}\Big| = \frac{x^{s-1}|\log x|}{e^x - 1} \leq g(x) := \begin{cases} C_0C_1x^{a-\varepsilon-2} & \text{if } x \in (0, 1) \\ n!x^{b-1-n} & \text{if } x \geq 1 \end{cases}$$

for all $s \in [a, b]$. Since $g \in \mathcal{L}^1((0, \infty), \mathbb{R})$ the theorem on the differentiation of parameter integrals applies.

Extra questions for further practice

4. Let μ be a measure defined on the σ-algebra $\mathcal{A}$ of subsets of X. Supposse that $\mu(X) < \infty$ and let $1 \leq p < q < \infty$.

(a) Show that $\|f\|_p \leq \mu(X)^{1/p-1/q}\|f\|_q$, that is $L_q(X,\mathbb{C}) \subseteq L_p(X,\mathbb{C})$.

Solution: We use Hölder's inequality writing $|f|^p = 1 \cdot |f|^p$. To get the L^q-norm we choose one exponend to be $r = \frac{q}{p} > 1$ and compute the other by solving the equation $1/r + 1/s = 1$. I f we do that we get $s = \frac{q}{q-p}$. Hence by Hölder's inequality

$$\|f\|_p^p = \int_X 1 \cdot |f|^p \, d\mu \leq \Big(\int_X 1^{\frac{q}{q-p}} \, d\mu\Big)^{\frac{q-p}{q}} \Big(\int_X |f|^{p\frac{q}{p}} \, d\mu\Big)^{\frac{p}{q}} = \mu(X)^{1-\frac{p}{q}}\|f\|_q^p.$$

Taking the above to the $1/p$-th power we get the required inequality.

(b) Give an example to show that the above is a proper inclusion if $1 \leq p < q < \infty$. In particular the inequality implies that every function in $L_q(X,\mathbb{C})$ is also in $L_p(X,\mathbb{C})$.

Solution: Let $1 \leq p < q$ and choose $\alpha > 0$ such that $1/q < \alpha < 1/p$. Set $f(x) := x^{-\alpha}$ for $x \in (0,1]$ and zero otherwise. Then

$$\|f\|_p = \int_0^1 x^{-\alpha p} \, dx < \infty$$

since $\alpha p < 1$ but

$$\|f\|_q = \int_0^1 x^{-\alpha q} \, dx = \infty$$

since $\alpha q > 1$ (explicit calculation).

(c) Give an example to show that $L_q(X,\mathbb{C}) \subseteq L_p(X,\mathbb{C})$ is not true in general if $\mu(X) = \infty$.

Solution: Let $1 \leq p < q$ and choose $\alpha > 0$ such that $1/q < \alpha < 1/p$. Set $f(x) := x^{-\alpha}$ for $x \in [1,\infty]$ and zero otherwise. Then

$$\|f\|_p = \int_1^\infty x^{-\alpha p} \, dx = \infty$$

since $\alpha p < 1$ but

$$\|f\|_q = \int_1^\infty x^{-\alpha q} \, dx < \infty$$

since $\alpha q > 1$ (explicit calculation).

5. Let μ be a measure on the σ-algebra $\mathcal{A}$ of subsets of X. If $f\colon X \to \mathbb{C}$ is a measurable function we define

$$\|f\|_\infty := \operatorname*{ess-sup}_{x\in\mathbb{X}} |f(x)| := \inf\big\{t > 0\colon \mu(\{x \in X\colon |f(x)| > t\}) = 0\big\}.$$

We call this the *essential supremum* of $|f|$ and set

$$\mathcal{L}^\infty(X,\mathbb{C}) := \{f\colon X \to \mathbb{C} \mid f \text{ measurable and } \|f\|_\infty < \infty\}.$$

(a) Let $f \in \mathcal{L}^\infty(X,\mathbb{C})$. Show that $N := \{x \in X\colon |f(x)| > \|f\|_\infty\}$ has zero measure, and that $A_\varepsilon := \{x \in X\colon |f(x)| > \|f\|_\infty - \varepsilon\}$ has positive measure for every $\varepsilon > 0$. Hence explain the term "essential supremum".

Solution: For every $n \in \mathbb{N}$ we set

$$N_n := \{x \in X\colon |f(x)| > \|f\|_\infty + 1/n\}$$

Then by definition of the essential supremum $\mu(N_n) = 0$ for all $n \in \mathbb{N}$. Clearly $\bigcup_{n\in\mathbb{N}} N_n = N$ and $N_n \subseteq N_{n+1}$ for all $n \in \mathbb{N}$. By the monotonicity properties of measures

$$\mu(N) = \mu\Big(\bigcup_{n\in\mathbb{N}} N_n\Big) = \lim_{n\to\infty} \mu(N_n) = 0$$

as claimed. By definition of $\|f\|_\infty$ we have $\mu(A_\varepsilon) > 0$ for all $\varepsilon > 0$. Hence f is larger than $\|f\|_\infty$ only on a set of measure zero. Moreover, f is larger than any number smaller than $\|f\|_\infty$ on a "substantially larger" set, namely a set of positive measure. In particular, modifying f on a set of measure zero preserves the essential supremum, so it is the measure theoretic supremum of a function.

(b) Let N be the set from (a) and M a set of zero measure with $N \subset M$. Show that $\sup_{x\in X\setminus M} |f(x)| = \text{ess-sup}_{x\in X} |f(x)|$.

Solution: From the definition of N it is clear that $|f(x)| \leq \|f\|_\infty$, that is,

$$\sup_{x\in X\setminus M} |f(x)| \leq \operatorname*{ess-sup}_{x\in X} |f(x)| = \|f\|_\infty.$$

Clearly $\mu(A_\varepsilon \cap M^c) = \mu(A_\varepsilon) > 0$ for all $\varepsilon > 0$. Hence, if we fix $\varepsilon >$, then there exists $x \in X \setminus M$ such that $|f(x)| > \|f\|_\infty - \varepsilon$ and therefore $\sup_{x\in X\setminus M} |f(x)| \geq \|f\|_\infty - \varepsilon$. Since this is true for all $\varepsilon > 0$ we get

$$\sup_{x\in X\setminus M} |f(x)| \geq \operatorname*{ess-sup}_{x\in X} |f(x)| = \|f\|_\infty.$$

(c) For $f, g \in \mathcal{L}^\infty(X, \mathbb{C})$ and $\alpha \in \mathbb{C}$ prove the following:

(i) Show that $\|f\|_\infty = 0$ if and only if $f(x) = 0$ almost everywhere.

Solution: By (a) the set $\{x \in x\colon |f(x)| > \|f\|_\infty = 0\}$ has zero measure, so $f = 0$ almost everywhere.

(ii) $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$

Solution: The assertion is obvious for $\alpha = 0$, so assume that $\alpha \neq 0$. Then $|\alpha f(x)| > t$ if and only if $|f(x)| > s$ for $s = t/|\alpha|$. Hence

$$\begin{aligned}\|\alpha f\|_\infty &= \inf\big\{t > 0\colon \mu(\{x \in X\colon |\alpha f(x)| > t\}) = 0\big\} \\ &= \inf\big\{t > 0\colon \mu(\{x \in X\colon f(x)| > t/|\alpha|\}) = 0\big\} \\ &= \inf\big\{s|\alpha| > 0\colon \mu(\{x \in X\colon f(x)| > s\}) = 0\big\} \\ &= |\alpha| \inf\big\{s > 0\colon \mu(\{x \in X\colon |f(x)| > s\}) = 0\big\} = |\alpha|\|f\|_\infty\end{aligned}$$

as claimed.

(iii) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

Solution: We set $N_1 := \{x \in X\colon |f(x)| > 0\}$, $N_2 := \{x \in X\colon |g(x)| > 0\}$ and $N_3 := \{x \in X\colon |f(x) + g(x)| > 0\}$. Then let $M := N_1 \cup N_2 \cup N_3$. Then by (a) and (b) we have $\mu(M) = 0$ and

$$\begin{aligned}\|f\|_\infty &= \sup_{x\in X\setminus M} |f(x)| \\ \|g\|_\infty &= \sup_{x\in X\setminus M} |g(x)| \\ \|f+g\|_\infty &= \sup_{x\in X\setminus M} |f(x) + g(x)|.\end{aligned}$$

Hence

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

for all $x \in X \setminus M$. Taking the supremum on the left hand side we get

$$\|f + g\|_\infty = \sup_{x\in X\setminus M} |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

as claimed.

(d) Prove that Hölder's inequality remains true if $p = 1$ and $q = \infty$, that is,

$$\Big|\int_X fg\,d\mu\Big| \leq \|f\|_1\|g\|_\infty$$

for all $f \in \mathcal{L}^1(X,\mathbb{C})$ and $g \in \mathcal{L}^\infty(X,\mathbb{C})$.

Solution: By (b) we have

$$\Big|\int_X fg\,d\mu \leq \int_X |f||g|\,d\mu = \int_{X\setminus N} |f||g|\,d\mu \leq \int_{X\setminus N} |g| \sup_{x\in X\setminus N} |f(x)|\,d\mu = \|f\|_1\|g\|_\infty$$

6. Generalise Question **1** as follows: Let $p, p_k \in [1,\infty]$, $k = 1,\dots,n$, with $1/p = 1/p_1 + \cdots + 1/p_n$. Show that for measurable functions $f_1,\dots,f_n$

$$\|f_1 \dots f_n\|_p \leq \|f_1\|_{p_1}\|f_2\|_{p_2}\cdots\|f_n\|_{p_n}.$$

Solution: We use a proof by induction. If $n = 2$ the inequality is proved in Question **1**. Assume the assertion holds for $n-1$ functions. Define q by

$$\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}}.$$

Then clearly $1/p = 1/q + 1/p_n$ and so we can apply Question **1** and the induction hypothesis to get

$$\|f_1 \dots f_n\|_p \leq \|f_1 \dots f_{n-1}\|_q\|f_n\|_{p_n} \leq \big(\|f_1\|_{p_1}\|f_2\|_{p_2}\cdots\|f_{n-1}\|_{p_{n-1}}\big)\|f_n\|_{p_n}$$

as required.

7. Suppose that $f_n \to f$ in $L^p(\mathbb{R})$ with $p \in [1,\infty)$ and that (g_n) is a bounded sequence in $L^\infty(\mathbb{R})$ with $g_n \to g$ pointwise. Prove that $f_ng_n \to fg$ in $L^p(\mathbb{R})$.

Solution: By assumption there exists $M > 0$ such that $\|g_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Hence we have

$$\|f_ng_n - fg\|_p \leq \|(f_n - f)g_n\|_p + \|f(g_n - g)\|_p$$

for all $n \in \mathbb{N}$. First note that

$$\|(f_n - f)g_n\|_p \leq \|g_n\|_\infty\|f_n - f\|_p \leq M\|f_n - f\|_p \to 0$$

as $n \to \infty$. Second note that $|f(g_n - g)|^p \leq (2M)^p|f|^p$ almost everywhere. Moreover, $fg_n \to fg$ pointwise and $(2M)^p|f|^p \in L^1(\mathbb{R})$ and so by the dominated convergence theorem

$$\|f(g_n - g)\|_p^p = \int_{-\infty}^{\infty} |f(g_n - g)|^p\,dx \to 0.$$

Challenge questions (optional)

8. Let μ be a measure on the σ-algebra $\mathcal{A}$ of subsets of X. If $f\colon X \to \mathbb{C}$ and

$$\|f\|_\infty := \operatorname*{ess-sup}_{x\in\mathbb{X}} |f(x)| := \inf\big\{\alpha > 0\colon \mu\{x \in X\colon |f(x)| > \alpha\}) = 0\big\}.$$

Let (f_n) be a Cauchy sequence in $\mathcal{L}^\infty(X,\mathbb{C})$, that is, with respect to the (essential) supremum norm.

(a) Show that there exists a set $N \subset X$ with $\mu(N) = 0$, so that

$$\sup_{x \in X \setminus N} |f_n(x) - f_m(x)| = \operatorname*{ess-sup}_{x \in X} |f_n(x) - f_m(x)|$$

for all $n, m \in \mathbb{N}$.

Solution: For $n, m \in \mathbb{N}$ define the sets $A_{n,m} := \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$, where $\|f_n - f_m\|_\infty$ is the essential supremum norm. By Question **5**(a) we have $\mu(A_{n,m}) = 0$ and if we set $N := \bigcup_{n,m \in \mathbb{N}} A_{n,m}$ then also $\mu(N) = 0$. Now Question **5**(b) implies that

$$\sup_{x \in X \setminus N} |f_n(x) - f_m(x)| = \operatorname*{ess-sup}_{x \in X} |f_n(x) - f_m(x)|$$

for all $n, m \in \mathbb{N}$ for the above set N defined above.

(b) If N is the set from the previous part, show that $f_n \to f$ uniformly on $X \setminus N$ for some $f \in \mathcal{L}^\infty(X, \mathbb{C})$.

Solution: The sequence (f_n) is uniformly Cauchy on $X \setminus N$ and therefore by a result from analysis it converges uniformly to some bounded function f defined on $X \setminus N$. We define $f(x) := 0$ for $x \in N$. Then clearly $1_{X \setminus N} f_n \to f$ pointwise on X and since $1_{X \setminus N} f_n$ is measurable the pointwise limit f is measurable as well. Hence $f \in \mathcal{L}^\infty(X, \mathbb{C})$.

(c) Let $L^\infty(X, \mathbb{C}) := \{[f] : f \in \mathcal{L}^\infty(X, \mathbb{C})\}$, where $[f]$ is the equivalence class of f with respect to the equivalence relation given by $f \sim g$ if $f = g$ almost everywhere. Show that $L^\infty(X, \mathbb{C})$ is a complete normed space.

Solution: By Question **5**(c) $L^\infty(X, \mathbb{C})$ is a normed space. Let $[f_n]$ be a Cauchy sequence with $f_n \in \mathcal{L}^\infty(X, \mathbb{C})\}$. By the previous part there exists $f \in \mathcal{L}^\infty(X, \mathbb{C})$ with $\|f_n - f\|_\infty \to 0$. A similar assertion holds if $\tilde{f}_n \in [f_n]$ are different representatives of the equivalence class. Then there exists $\tilde{f}$ with $\|\tilde{f}_n - \tilde{f}\|_\infty \to 0$. Since $f_n = \tilde{f}_n$ almost everywhere we get

$$\|f - \tilde{f}\|_\infty \leq \|f - f_n\|_\infty + \|f_n - \tilde{f}\|_\infty = \|f - f_n\|_\infty + \|\tilde{f}_n - \tilde{f}\|_\infty \to 0$$

as $n \to \infty$. Hence $\|f - \tilde{f}\|_\infty = 0$ and therefore $f = \tilde{f}$ almost everywhere. Hence $f_n \to f$ (or more precisely $[f_n] \to [f]$ in $L^\infty(X, \mathbb{C})$.