Material covered

(1) Fubini’s theorem and applications
(2) Product measures
(3) Hölder’s inequality and $L^p$-spaces
(4) Differentiation of integrals with parameters

Outcomes

After completing this tutorial you should

(1) be able to understand and be able to apply Fubini’s theorem.
(2) have an idea about the construction of a product measure.
(3) work with Hölder’s inequality.
(4) apply the theorems on the differentiation of integrals with parameters.

Summary of essential material

There are two theorems on iterated integrals.

**Tonelli’s Theorem** (for non-negative functions)

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty]$ be measurable as a function on $\mathbb{R}^{n \times m}$. Then there exist sets $N \subseteq \mathbb{R}^n$ and $M \subseteq \mathbb{R}^m$ of measure zero such that

(i) $y \mapsto f(x, y)$ is measurable as a function on $\mathbb{R}^m$ for all $x \in \mathbb{R}^n \setminus N$;

(ii) $x \mapsto f(x, y)$ is measurable as a function on $\mathbb{R}^n$ for all $y \in \mathbb{R}^m \setminus M$.

If we modify $f$ on the set $N \times M$ of measure zero by setting $f(x, y) := 0$ for all $x \in N$ and all $y \in M$, then

(iii) $y \mapsto \int_{\mathbb{R}^n} f(x, y) \, dx$ is measurable as a function on $\mathbb{R}^m$;

(iv) $x \mapsto \int_{\mathbb{R}^m} f(x, y) \, dy$ is measurable as a function on $\mathbb{R}^n$.

Finally,

$$\int_{\mathbb{R}^{n \times m}} f(x, y) \, d(x, y) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} f(x, y) \, dy \right) \, dx = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} f(x, y) \, dx \right) \, dy \quad (1)$$
Questions to complete during the tutorial

1. Let \( f(x,y) = (x^2 - y^2)/(x^2 + y^2)^2 \) for \( 0 < x, y < 1 \). Show that
   \[
   \int_0^1 \left( \int_0^1 f(x,y) \, dy \right) \, dx = \frac{\pi}{4}, \quad \int_0^1 \left( \int_0^1 f(x,y) \, dx \right) \, dy = -\frac{\pi}{4}, \quad \int_0^1 \left( \int_0^1 |f(x,y)| \, dy \right) \, dx = \infty.
   \]

   **Solution:** From
   \[
   \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right),
   \]
   we have
   \[
   \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy = \left. \frac{y}{x^2 + y^2} \right|_{y=0}^{y=1} = \frac{1}{x^2 + 1}.
   \]
   Thus
   \[
   \int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = \int_0^1 \frac{1}{x^2 + 1} \, dx = \tan^{-1}(x) \bigg|_{x=0}^{x=1} = \frac{\pi}{4}.
   \]

   Similarly,
   \[
   \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial x} \left( \frac{-x}{x^2 + y^2} \right)
   \]
   implies that
   \[
   \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx = \left. \frac{-x}{x^2 + y^2} \right|_{x=0}^{x=1} = \frac{-1}{1 + y^2},
   \]
   and so
   \[
   \int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = \int_0^1 \frac{-1}{1 + y^2} \, dy = -\tan^{-1}(y) \bigg|_{y=0}^{y=1} = -\frac{\pi}{4}.
   \]
We know that
\[ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, y \, dx = \infty, \]
because otherwise Fubini’s Theorem would be contradicted. But let us verify this directly:
\[ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, y \, dx = \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, y \, dx + \int_1^x \frac{y^2 - x^2}{(x^2 + y^2)^2} \, y \, dy \]
\[ = \int_0^x \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) \, dy + \int_1^x \frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) \, dy \]
\[ = \frac{y}{x^2 + y^2} \bigg|_{y=0}^{y=x} + \frac{-y}{x^2 + y^2} \bigg|_{y=1}^{y=x} \]
\[ = \frac{1}{2x} + \left( \frac{1}{2x} - \frac{1}{x^2 + 1} \right) \]
\[ = \frac{1}{x} - \frac{1}{1 + x^2}. \]

Clearly
\[ \int_0^1 \frac{1}{x} \, dx = \infty \quad \text{and} \quad \int_0^1 \frac{1}{1 + x^2} \, dx < \infty, \]
and so
\[ \int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, y \, dx \right) \, dy = \infty. \]

2. For \( t > 0 \) consider \( f(t) := \left( \int_0^t e^{-x^2} \, dx \right)^2 + \int_0^1 \frac{e^{-t^2(1+x^2)}}{1 + x^2} \, dx. \)

(a) By differentiation, show that \( f \) is constant. First complete the calculation, then justify the steps by using the theorems on the differentiation of integrals with parameters.

\[ \textbf{Solution:} \quad \text{We show that } f \text{ is differentiable with zero derivative. By the fundamental theorem of calculus} \]
\[ \frac{d}{dt} \int_0^t e^{-x^2} \, dx = e^{-t^2}, \]
so by the chain rule and the substitution \( y = x/t \)
\[ \frac{d}{dt} \left( \int_0^t e^{-x^2} \, dx \right)^2 = 2e^{-t^2} \int_0^t e^{-x^2} \, dx = 2te^{-t^2} \int_0^1 e^{-t^2 y^2} \, dy = 2t \int_0^1 e^{-t^2(1+x^2)} \, dx. \]
The other integral is a parameter integral. Assuming that we can differentiate under the integral we get
\[ \frac{d}{dt} \int_0^1 \frac{e^{-t^2(1+x^2)}}{1 + x^2} \, dx = \int_0^1 \frac{\partial}{\partial t} \frac{e^{-t^2(1+x^2)}}{1 + x^2} \, dx = -2t \int_0^1 e^{-t^2(1+x^2)} \, dx \]
and therefore
\[ f''(t) = 2t \int_0^1 e^{-t^2(1+x^2)} \, dx - 2t \int_0^1 e^{-t^2(1+x^2)} \, dx = 0 \]
for all \( t > 0 \). We finally need to check whether we can differentiate under the integral sign. We note that
\[ \left| \frac{\partial}{\partial t} \frac{e^{-t^2(1+x^2)}}{1 + x^2} \right| = 2te^{-t^2(1+x^2)} \leq 2T \]
for all \( t \in [0, T] \). Since \( \int_0^T 2T \, dx = 2T < \infty \) the theorem on the differentiation of parameter integrals applies, showing that the integral is differentiable for \( t \in (0, T) \). Since this works for all \( T > 0 \) we conclude that \( f \) is differentiable for all \( t > 0 \).

(b) By looking at \( f(t) \) as \( t \to 0 \) and \( t \to \infty \) deduce that \( \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi}/2 \).

**Solution:** Since \( f \) is constant we have

\[
\lim_{t \to 0} f(t) = \lim_{t \to \infty} f(t).
\]

We compute the two limits. Clearly

\[
\int_0^t e^{-x^2} \, dx \to \int_0^0 e^{-x^2} \, dx = 0
\]

as \( t \to 0 \) and

\[
\int_0^t e^{-x^2} \, dx \to \int_0^\infty e^{-x^2} \, dx
\]

as \( t \to \infty \). Now

\[
|\frac{e^{-r^2(1+x^2)}}{1+x^2}| \leq \frac{1}{1+x^2}
\]

for all \( x \in [0, 1] \) and all \( t \geq 0 \). Since the right hand side has a finite integral over \([0, 1]\) we can apply the theorem on the continuity of parameter integrals and conclude that

\[
\int_0^1 \frac{e^{-r^2(1+x^2)}}{1+x^2} \, dx \int_0^1 \frac{1}{1+x^2} \, dx = \tan^{-1} x \bigg|_0^1 = \frac{\pi}{4}
\]

as \( t \to 0 \) and that

\[
\int_0^1 \frac{e^{-r^2(1+x^2)}}{1+x^2} \, dx \to \int_0^1 \frac{0}{1+x^2} \, dx = 0
\]

as \( t \to \infty \). Combining everything we see that

\[
\frac{\pi}{2} \lim_{t \to 0} f(t) = \lim_{t \to \infty} f(t) = \left( \int_0^\infty e^{-x^2} \, dx \right)^2.
\]

Taking square roots on both sides our claim follows.

(c) Use (b) and Fubini’s Theorem to show that \( \int_{\mathbb{R}^N} e^{-|x|^2} \, dx = \pi^{N/2} \), where \( |x| \) is the Euclidean norm of \( x \).

**Solution:** We give a proof by induction. For \( N = 1 \) we have

\[
\int_{\mathbb{R}} e^{-x^2} \, dx = 2 \int_0^\infty e^{-x^2} \, dx = 2 \sqrt{\frac{\pi}{2}} = \pi^{1/2}.
\]

Suppose now that the assertion is true for \( N - 1 \), that is,

\[
\int_{\mathbb{R}^{N-1}} e^{-|x|^2} \, dx = \pi^{(N-1)/2}.
\]

If \( x \in \mathbb{R}^N \), then we can write \( x = (y, z) \) with \( y \in \mathbb{R}^{N-1} \) and \( z \in \mathbb{R} \). Then \( |x|^2 = |y|^2 + z^2 \) by definition of the Euclidean norm. Hence by Fubini’s (or more specifically Tonelli’s theorem) we have

\[
\int_{\mathbb{R}^N} e^{-|x|^2} \, dx = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} e^{-|y|^2-z^2} \, dz \, dy
\]

\[
= \int_{\mathbb{R}^{N-1}} e^{-|y|^2} \int_{\mathbb{R}} e^{-z^2} \, dz \, dy = \left( \int_{\mathbb{R}^{N-1}} e^{-|y|^2} \, dy \right) \left( \int_{\mathbb{R}} e^{-z^2} \, dz \right).
\]
Using the induction assumption and the case \( N = 1 \) we get
\[
\int_{\mathbb{R}^N} e^{-|x|^2} \, dx = \pi^{(N-1)/2} \pi^{1/2} = \pi^{N/2}
\]
as claimed.

3. Let \( f \in L_p(X, \mathbb{C}) \cap L_q(X, \mathbb{C}) \) with \( 1 \leq p \leq q \leq \infty \), and let \( p < r < q \).

(a) Use Hölder’s inequality to prove the interpolation inequality
\[
\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}
\]
if \( \theta \in (0, 1) \) is chosen such that \( 1/r = \theta/p + (1 - \theta)/q \). In particular the inequality shows that \( L^p(X) \cap L^1(X) \subseteq L^r(X) \) for all \( r \in [p, q] \).

**Solution:** First assume that \( 1 \leq p < r < q = \infty \). Then
\[
\left( \int_X |f|^r \, d\mu \right)^{1/r} = \left( \int_X |f|^p |f|^{r-p} \, d\mu \right)^{1/r} \leq \left( \|f\|_p^{r-p} \int_X |f|^p \, d\mu \right)^{1/r} = \|f\|_p^{r/p} \|f\|_\infty^{1-p/r}.
\]
Hence \( \theta = p/r \) so that \( 1/r = \theta/p + (1 - \theta)/\infty \). If \( 1 \leq p < r < q < \infty \) we use the generalised Hölder inequality from Tutorials. From the assumptions we have
\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q - \theta}.
\]
Hence by the definition of the \( L^p \)-norms
\[
\|f\|_r = \|f\|^{\theta} \|f\|^{1-\theta} \leq \|f\|_p^{\theta} \|f\|_q^{1-\theta} = \|f\|_p^{\theta} \|f\|_q^{1-\theta}
\]
as claimed.

(b) Use the previous part and Young’s inequality to show that for every \( \varepsilon > 0 \) there exists a constant \( c(\varepsilon) > 0 \) such that
\[
\|f\|_r \leq \varepsilon \|f\|_p + c(\varepsilon) \|f\|_q.
\]

**Solution:** We use Young’s inequality in the form \( st \leq \theta t^{1/\theta} + (1 - \theta)s^{1/(1-\theta)} \) if \( s, t \geq 0 \) and \( \theta \in (0, 1) \), which we obtain from the inequality in lectures by taking the reciprocals of the exponents. Fix \( \varepsilon > 0 \) and apply that inequality to the inequality from the previous part:
\[
\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta} = \left( \varepsilon \|f\|_p \right)^\theta \left( \frac{\theta}{\varepsilon} \right)^{\theta - 1} \|f\|_q^{1-\theta} \leq \theta \varepsilon \|f\|_p + (1 - \theta) \left( \frac{\theta}{\varepsilon} \right)^{\theta - 1} \|f\|_q = \varepsilon \|f\|_p + (1 - \theta) \left( \frac{\theta}{\varepsilon} \right)^{\theta - 1} \|f\|_q.
\]
Hence we have the required inequality with \( c(\varepsilon) = (1 - \theta) \left( \frac{\theta}{\varepsilon} \right)^{\theta - 1} \|. 
4. (a) Use Fubini’s theorem and the relation \( \frac{1}{x} = \int_0^\infty e^{-xy} \, dy \) to show that

\[
\lim_{a \to \infty} \int_0^a \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
\]

**Solution:** by using the expression for \( 1/x \) above we have

\[
\int_0^a \frac{\sin x}{x} \, dx = \int_0^a \int_0^\infty e^{-xy} \, dx \, dy.
\]

To be able to interchange the order of integration we check the conditions of Fubini’s theorem. Using that \( |\sin x| \leq x \) for all \( x \geq 0 \) we get

\[
\int_0^a \int_0^\infty |e^{-xy} \sin x| \, dx \, dy \leq \int_0^a \int_0^\infty e^{-xy} \, dx \, dy = \int_0^a \frac{x}{x} \, dy = a
\]

for all \( a > 0 \). Hence we can interchange the integrals and get

\[
\int_0^a \frac{\sin x}{x} \, dx = \int_0^\infty \int_0^a e^{-xy} \, dx \, dy \, dx.
\]

To compute the inner integral note that \( \sin x = \text{Im} e^{ix} \). Now

\[
\int_0^a e^{-xy} e^{ix} \, dx = \int_0^a e^{x(i-y)} \, dy = \frac{1}{i-y} e^{x(i-y)} \bigg|_0^a = \frac{y+i}{1+y^2} (1 - e^a e^{-ay}).
\]

Hence

\[
\int_0^a e^{-xy} \sin x \, dy = \text{Im} \int_0^a e^{-xy} e^{ix} \, dy = \text{Im} \left( \frac{y+i}{1+y^2} (1 - e^a e^{-ay}) \right)
\]

\[
= \frac{1}{1+y^2} - \frac{e^{-ay}}{1+y^2} (\cos ax + y \sin ax).
\]

Alternatively we could compute the above integral by integrating by parts twice. Now clearly

\[
\frac{1}{1+y^2} - \frac{e^{-ay}}{1+y^2} (\cos ax + y \sin ax) \to 0
\]

as \( a \to \infty \) for all \( y > 0 \). Since \( ye^{-ay} \leq 1 \) for all \( a \geq 1 \) we have

\[
\left| \frac{e^{-ay}}{1+y^2} (\cos ax + y \sin ax) \right| \leq \frac{2}{1+y^2}
\]

for all \( y \geq 0 \) and all \( a \geq 1 \). Since \( 1/(1+y^2) \) is integrable we can apply the theorem on the continuity of parameter integrals to conclude that

\[
\lim_{a \to \infty} \int_0^a \frac{\sin x}{x} \, dx = \lim_{a \to \infty} \int_0^\infty \left( \frac{1}{1+y^2} - \frac{e^{-ay}}{1+y^2} (\cos ax + y \sin ax) \right) \, dy
\]

\[
= \int_0^\infty \frac{1}{1+y^2} \, dy = \tan^{-1} y \bigg|_0^\infty = \frac{\pi}{2}
\]

as claimed.
(b) Prove that $\frac{\sin x}{x} \not\in L^1((0, \infty), \mathbb{R})$.

**Solution:** We can write
\[
\int_0^\infty \frac{|\sin x|}{x} \, dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} \, dx \\
\geq \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{(n+1)\pi} \, dx = \frac{1}{\pi} \int_0^\pi \sin x \, dx \sum_{n=1}^\infty \frac{1}{n}.
\]

Since the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges we have
\[
\int_0^\infty \frac{|\sin x|}{x} \, dx = \infty
\]
as claimed.

5. Show that
\[
\int_0^\infty e^{-\gamma} \sin^2(y) \frac{dy}{y} = \frac{\log 5}{4}.
\]
by applying Fubini’s theorem, looking at the integrand as an integral of $f(x, y) = \sin(2xy)e^{-\gamma}$.

**Solution:** Consider
\[
\int_0^\infty \left( \int_0^1 \sin(2xy)e^{-\gamma} \, dx \right) \, dy.
\]
The inner integral equals
\[
e^{-\gamma} \int_0^1 \sin(2xy) \, dx = e^{-\gamma} \frac{-\cos(2xy)}{2y} \bigg|_{x=0}^{x=1} = e^{-\gamma} \frac{1 - \cos(2y)}{2y} = e^{-\gamma} \frac{\sin^2(y)}{y}.
\]
Let $I(x) = \int_0^\infty \sin(2xy)e^{-\gamma} \, dy$. Two integrations by parts yield
\[
I(x) = 2x - 4x^2 \int_0^\infty \sin(2xy)e^{-\gamma} \, dy,
\]
so that $I(x) = 2x/(1 + 4x^2)$. Thus
\[
\int_0^1 \left( \int_0^\infty \sin(2xy)e^{-\gamma} \, dy \right) \, dx = \int_0^1 \frac{2x}{1 + 4x^2} \, dx = \frac{1}{4} \log(1 + 4x^2) \bigg|_{x=0}^{x=1} = \frac{\log 5}{4}.
\]
Fubini’s Theorem is applicable because
\[
\int_0^\infty \left( \int_0^1 \left| \sin(2xy)e^{-\gamma} \right| \, dx \right) \, dy \leq \int_0^\infty \left( \int_0^1 e^{-\gamma} \, dx \right) \, dy = \int_0^\infty e^{-\gamma} \, dy = 1 < \infty.
\]
Thus
\[
\int_0^\infty e^{-\gamma} \frac{\sin^2(y)}{y} \, dy = \frac{\log 5}{4}.
\]

6. Use the dominated convergence theorem to show that
\[
\lim_{n \to \infty} \int_0^\infty f_n(x) \, dx = \int_0^\infty \frac{x}{e^x - 1} \, dx, \quad \text{where} \quad f_n(x) := \frac{n}{e^x - 1} \sin \frac{x}{n}.
\]
for all \( x > 0 \).

**Solution:** First note that
\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x \sin \frac{x}{n}}{e^x - 1} = \frac{x}{e^x - 1}
\]
for every \( x > 0 \). We find an integrable dominating function. Note that \( \left| \frac{\sin y}{y} \right| \leq 1 \) for all \( y \in \mathbb{R} \) and that exponential series
\[
e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{3!} \geq \frac{x^k}{k!}
\]
for all \( x > 0 \) and all \( k \geq 1 \). Hence,
\[
|f_n(x)| = \frac{x}{e^x - 1} \left| \frac{\sin \frac{x}{n}}{\frac{x}{n}} \right| \leq \frac{k!}{x^{k-1}}
\]
for all \( x > 0 \) and all \( k \geq 1 \). Choosing \( k = 1 \) and \( k = 3 \) we see that
\[
|f_n(x)| \leq g(x) := \min\{1, \frac{6}{x^2}\}.
\]
Since
\[
\int_0^\infty g(x) \, dx \leq 1 + 6 \int_1^\infty \frac{1}{x^2} \, dx = 1 + \frac{1}{3} = \frac{4}{3} < \infty
\]
the dominated convergence theorem implies the convergence of the integrals.

7. Show that the function \( F : (0, \infty) \to \mathbb{R} \) given by
\[
F(t) := \int_{\mathbb{R}} e^{-tx} \cos x \, dx
\]
is differentiable with respect to \( t \in (0, \infty) \).

**Solution:** First note that for every \( t > 0 \) we have \( |e^{-tx^2} \cos x| \leq e^{-tx^2} \). Moreover, using the exponential series
\[
e^{tx^2} = 1 + tx^2 + \frac{t^2 x^4}{2} + \cdots \geq 1 + tx^2
\]
and hence
\[
e^{-tx^2} \leq \frac{1}{1 + tx^2} \quad \text{and} \quad x^2 e^{-tx^2} \leq \frac{x^2}{tx^2 + t^2 x^4/2} = \frac{1}{t} \frac{1}{1 + tx^2/2} \leq \frac{2}{t} \frac{1}{1 + tx^2}
\]
for all \( x \in \mathbb{R} \) and all \( t > 0 \). Now, using the substitution \( y = \sqrt{t} x \) we see that
\[
\int_{\mathbb{R}} \frac{1}{1 + tx^2} \, dx = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} \frac{1}{1 + y^2} \, dy = \left[ \tan^{-1} y \right]_{-\infty}^{\infty} = \frac{\pi}{\sqrt{t}}.
\]
In particular, \( e^{-tx^2} \cos x \) is integrable for all \( t > 0 \) (but not for \( t = 0 \)). If \( a > 0 \), then
\[
\left| \frac{d}{dt} e^{-tx^2} \cos x \right| = \left| -x^2 e^{-tx^2} \cos x \right| \leq x^2 e^{-tx^2} \leq \frac{2}{t} \frac{1}{1 + tx^2} \leq \frac{2}{a} \frac{1}{1 + tx^2} \in L^1(\mathbb{R})
\]
for all \( t \geq a \) and all \( x \in \mathbb{R} \). Hence, the theorem on the differentiability of integrals with parameters implies that \( F \) is differentiable on \([a, \infty)\) for all \( a > 0 \). Hence it is differentiable on \((0, \infty)\).

**Note:** 1. It is really sufficient to state that \( x^2 e^{-ax^2} \) is continuous on \( \mathbb{R} \) and exponentially decaying, and therefore integrable.
8. Let \( A \) be a \( \sigma \)-algebra of subsets of \( X \) and \( B \) a \( \sigma \)-algebra of subsets of \( Y \). Denote by \( A \times B \) the smallest \( \sigma \)-algebra of subsets of \( X \times Y \) such that it contains all generalised rectangles \( A \times B \) with \( A \in A \) and \( B \in B \). Let \( C \in A \times B \). For \( x \in X \) set \( C_x := \{ y \in Y : (x, y) \in C \} \) and for \( y \in Y \) set \( C^y := \{ x \in X : (x, y) \in C \} \). Define \( S := \{ C \in A \times B : C_x \in B \text{ for all } x \in X \} \)

(a) Show that every measurable generalised rectangle in \( A \times B \) is in \( S \).

Solution: Let \( C = A \times B \) with \( A \in A \) and \( B \in B \). Then
\[
C_x = \begin{cases} 
B & \text{if } x \in A \\
\emptyset & \text{if } x \notin A.
\end{cases}
\]

Since \( B \) and \( \emptyset \) are measurable \( C \in S \).

(b) Show that \( S \) is a \( \sigma \)-algebra.

Solution: We prove the three properties of a \( \sigma \)-algebra.

(i) Clearly \( \emptyset = \emptyset \times \emptyset \in S \).

(ii) Suppose now that \( C \in S \) and fix \( x \in X \). If \( y \in C^c_x \), then \( (x, y) \notin C \) and so \( (x, y) \notin C^c \). Hence \( y \in (C^c)_x \). On the other hand, if \( y \in (C^c)_x \), then \( (x, y) \in C^c \) and so \( (x, y) \notin C \). Hence \( y \notin C_x \), that is, \( y \in C^c_x \). Therefore \( (C^c)_x = C^c \in B \) since \( B \) is a \( \sigma \)-algebra.

(iii) Suppose now that \( C_k \in S \) for all \( k \in \mathbb{N} \). If \( y \in (\bigcup_{k \in \mathbb{N}} C_k)_x \), then \( (x, y) \in \bigcup_{k \in \mathbb{N}} C_k \). Hence there exists \( k_y \) with \( (x, y) \in C_{k_y} \) and so \( y \in (C_{k_y})_x \). This shows that \( (\bigcup_{k \in \mathbb{N}} C_k)_x \subseteq \bigcup_{k \in \mathbb{N}} (C_k)_x \). If \( y \in \bigcup_{k \in \mathbb{N}} (C_k)_x \), then there exists \( k_0 \) such that \( y \in (C_{k_0})_x \). But then \( (x, y) \in C_{k_0} \) and so \( (x, y) \in \bigcup_{k \in \mathbb{N}} C_k \). Hence \( \bigcup_{k \in \mathbb{N}} (C_k)_x \subseteq (\bigcup_{k \in \mathbb{N}} C_k)_x \). By putting together both we get
\[
\bigcup_{k \in \mathbb{N}} (C_k)_x = (\bigcup_{k \in \mathbb{N}} C_k)_x \in B
\]
since \( (C_k)_x \in B \) for all \( k \in \mathbb{N} \) and \( B \) is a \( \sigma \)-algebra.

(c) Show that \( C_x \in B \) and \( C^y \in A \) whenever \( C \in A \times B \).

Solution: Since \( S \) is a \( \sigma \)-algebra containing all generalised measurable rectangles and \( A \times B \) is the \( \sigma \)-algebra generated by the measurable rectangles we have \( A \times B \subseteq S \).

Interchanging the roles of \( x \) and \( y \) also \( S' := \{ C \in A \times B : C^y \in A \} \) is a \( \sigma \)-algebra. A similar argument shows that \( A \times B \subseteq S' \). Hence
\[
A \times B \subseteq S \cap S'.
\]

This implies that \( C_x \in B \) and \( C^y \in A \) whenever \( C \in A \times B \).

Challenge questions (optional)

9. By a complete measure \( \mu \) we mean a measure such that if \( M \) is measurable and \( \mu(M) = 0 \), then every subset of \( M \) is measurable (and has measure zero). The Lebesgue measure defined on the Lebesgue \( \sigma \)-algebra is complete. It is not complete when only defined on the Borel \( \sigma \)-algebra.

Suppose that \( \mu \) is a complete measure defined on the \( \sigma \)-algebra \( A \) of subsets of \( X \). Let \( f : X \to \mathbb{R} \) be a measurable function and \( g : X \to \mathbb{R} \) a function with \( f = g \) almost everywhere. Prove that \( g \) is measurable. Is this still true if \( \mu \) is not complete?

Solution: By assumption there exists a measurable set \( N \subseteq X \) with \( \mu(X) = 0 \) such that \( f(x) = g(x) \) for all \( x \in X \setminus N \). Mow consider the set
\[
N_0 := \{ x \in X : f(x) \neq g(x) \}
\]
Then clearly $N_0 \subseteq N$ and by the completeness of $\mu$ the set $N_0$ is measurable. Now let $U \subseteq \mathbb{R}$ be open. If $0 \notin U$, then
\[
\{ x \in X : g(x) - f(x) \in U \} \subseteq N_0
\]
is measurable because of the completeness of $\mu$. Similarly
\[
\{ x \in X : g(x) - f(x) = 0 \} = X \setminus N_0
\]
is measurable. Hence if $U$ is open and $0 \in U$, then $U \setminus \{0\}$ is open and so
\[
\{ x \in X : g(x) - f(x) \in U \} = \{ x \in X : g(x) - f(x) \in U \setminus \{0\} \} \cup \{ x \in X : g(x) - f(x) = 0 \}
\]
is measurable. Hence $g - f$ is a measurable function. Hence $g = g + (f - g)$ is measurable because it is the sum of two measurable functions.

Assume now that $\mu$ is not complete. Then there exist sets $N_0, N$ such that $N_0 \subseteq N$, $N$ is measurable, but $N_0$ is not measurable. Then the constant function $f = 0$ is measurable and $g := 1_{N_0}$ is not measurable. However, $f = g$ almost everywhere. Hence if $f$ is measurable and $f = g$ almost everywhere, then $g$ does not need to be measurable.

10. Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces. For arbitrary sets $C \subseteq X \times Y$ define
\[
\lambda^*(C) := \inf \left\{ \sum_{k=0}^{\infty} \mu(A_k) \nu(B_k) : A_k \in \mathcal{A}, B_k \in \mathcal{B} \text{ for all } k \in \mathbb{N} \text{ and } C \subseteq \bigcup_{k=0}^{\infty} (A_k \times B_k) \right\}
\]
(a) Show that $\lambda^*$ is an outer measure on the product space $X \times Y$. (Look at the proof that the Lebesgue outer measure is an outer measure and mimic the proof).

Solution: (i) Take the trivial sets $A_k = \emptyset$ and $B_k = \emptyset$. Then $\emptyset \subseteq \bigcup_{k=0}^{\infty} (A_k \times B_k)$ and so
\[
0 \leq \lambda^*(\emptyset) \leq \sum_{k=0}^{\infty} \mu(A_k) \nu(B_k) = 0
\]
since $\mu(A_k) = 0$ and $\nu(B_k) = 0$ for all $k \in \mathbb{N}$. Hence $\lambda^*(\emptyset) = 0$.

(ii) Let $C, C_k \subseteq \mathbb{R}$, $k \in \mathbb{N}$ and $C \subseteq \bigcup_{k=0}^{\infty} C_k$. Let $\varepsilon > 0$. By definition of an infimum we can choose sets $A_{jk} \times B_{jk}$ such that $A_{jk} \in \mathcal{A}$, $B_{jk} \in \mathcal{B}$, $C_j \subseteq \bigcup_{j=0}^{\infty} A_{jk} \times B_{jk}$ and
\[
\lambda^*(C_k) + \frac{\varepsilon}{2^{k+1}} > \sum_{j=0}^{\infty} \mu(A_{jk}) \nu(B_{jk}).
\]
Then
\[
C \subseteq \bigcup_{k=0}^{\infty} C_k \subseteq \bigcup_{k=0}^{\infty} \bigcup_{j=1}^{\infty} A_{jk} \times B_{jk}
\]
Since $\mathbb{N} \times \mathbb{N}$ is countable, the above is a cover of $C$ with countably many open rectangles. Hence by definition of $\lambda^*$, that is, the definition of an infimum
\[
\lambda^*(C) \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \mu(A_{jk}) \nu(B_{jk}) < \sum_{k=0}^{\infty} \left( \lambda^*(C_k) + \frac{\varepsilon}{2^{k+1}} \right) = \sum_{k=0}^{\infty} \lambda^*(C_k) + \varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary we see that
\[
\lambda^*(C) \leq \sum_{k=0}^{\infty} \lambda^*(C_k)
\]
as claimed.
(b) Suppose that \( C = A \times B \) for given sets \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \).

(i) Show that \( \lambda^*(A \times B) \leq \mu(A)v(B) \).

**Solution:** If \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), then clearly
\[
A \times B \subseteq (A \times B) \cup (\emptyset \times \emptyset) \cup (\emptyset \times B) \cup \ldots
\]
and hence from definition of \( \lambda^* \) as an infimum over all covers we conclude that
\[
\lambda^*(A \times B) \leq \mu(A)v(B) + \mu(\emptyset)v(\emptyset) + \mu(\emptyset)v(\emptyset) + \cdots = \mu(A)v(B).
\]

(ii) Let \( A_k \in \mathcal{A} \) and \( B_k \in \mathcal{B} \), \( k \in \mathbb{N} \) be such that and \( A \times B \subseteq \bigcup_{k=0}^{\infty} (A_k \times B_k) \). Then clearly
\[
1_{A \times B}(x, y) = 1_A(x)1_B(y) \leq \sum_{k=0}^{\infty} 1_{A_k \times B_k}(x, y) = \sum_{k=0}^{\infty} 1_{A_k}(x)1_{B_k}(y). \tag{4}
\]
Use the above inequality and the monotone convergence theorem to show that
\[
\mu(A)v(B) = \int_X 1_A \, d\mu \int_Y 1_B \, d\nu \leq \sum_{k=0}^{\infty} \mu(A_k)v(B_k). \tag{5}
\]

**Solution:** Integrating the inequality (4) with respect to \( \mu \) over \( X \) we obtain
\[
\mu(A)1_B(y) \leq \int_X \sum_{k=0}^{n} 1_{A_k}(x)1_{B_k}(y) \, d\mu(x)
\]
for every fixed \( y \in Y \). As the series has positive terms, for every fixed \( y \) the partial sums
\[
\lim_{n \to \infty} \sum_{k=0}^{n} 1_{A_k}(x)1_{B_k}(y)
\]
form a monotone increasing sequence of measurable functions in \( x \). Hence the monotone convergence theorem implies that for every fixed \( y \in Y \)
\[
\int_X \sum_{k=0}^{\infty} 1_{A_k}(x)1_{B_k}(y) \, d\mu(x) = \lim_{n \to \infty} \int_X \sum_{k=0}^{n} 1_{A_k}(x)1_{B_k}(y) \, d\mu(x)
\]
\[
= \lim_{n \to \infty} \sum_{k=0}^{n} \int_X 1_{A_k}(x) \, d\mu(x)1_{B_k}(y) = \sum_{k=0}^{\infty} \int_X 1_{A_k}(x) \, d\mu(x)1_{B_k}(y) = \sum_{k=0}^{\infty} \mu(A_k)1_{B_k}(y)
\]
Combining this with (5) we see that for every fixed \( y \in Y \),
\[
\mu(A)1_B(y) \leq \sum_{k=0}^{\infty} \mu(A_k)1_{B_k}(y) \tag{6}
\]
Repeating the above argument using the monotone convergence theorem we obtain
\[
\mu(A)v(B) = \int_Y \mu(A)1_B(y) \, d\nu(y) \leq \int_Y \sum_{k=0}^{\infty} \mu(A_k)1_{B_k}(y) \, d\nu(y)
\]
\[
= \sum_{k=0}^{\infty} \mu(A_k) \int_Y 1_{B_k}(y) \, d\nu(y) = \sum_{k=0}^{\infty} \mu(A_k)v(B_k),
\]
proving the claim.
(iii) Hence show that \( \lambda^*(A \times B) = \mu(A)\nu(B) \).

**Solution:** By definition of an infimum, part (ii) implies that \( \mu(A)\nu(B) \leq \lambda^*(A \times B) \). We know from the previous part (i) we know the converse inequality \( \lambda^*(A \times B) \leq \mu(A)\nu(B) \). Hence, we have equality.