The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 7 (Week 8)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) Fubini's theorem and applications

(2) Product measures

Outcomes

After completing this tutorial you should

(1) be able to understand and be able to appy Fubini's theorem.

(2) have an idea about the construction of a product measure.

Questions to complete during the tutorial

1. Let $f(x,y) = (x^2 - y^2)/(x^2 + y^2)^2$ for $0 < x, y < 1$. Show that

$$\int_0^1 \Bigl(\int_0^1 f(x,y)\,dy\Bigr)\,dx = \frac{\pi}{4},\quad \int_0^1 \Bigl(\int_0^1 f(x,y)\,dx\Bigr)\,dy = -\frac{\pi}{4} \text{ and } \int_0^1 \Bigl(\int_0^1 |f(x,y)|\,dy\Bigr)\,dx = \infty.$$

Solution: From

$$\frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{\partial}{\partial y}\Bigl(\frac{y}{x^2+y^2}\Bigr),$$

we have

$$\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dy = \frac{y}{x^2+y^2}\Big|_{y=0}^{y=1} = \frac{1}{x^2+1}.$$

Thus

$$\int_0^1 \Bigl(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dy\Bigr)\,dx = \int_0^1 \frac{1}{x^2+1}\,dx = \tan^{-1}(x)\Big|_{x=0}^{x=1} = \frac{\pi}{4}.$$

Similarly,

$$\frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\Bigl(\frac{-x}{x^2+y^2}\Bigr)$$

implies that

$$\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dx = \frac{-x}{x^2+y^2}\Big|_{x=0}^{x=1} = \frac{-1}{1+y^2},$$

and so

$$\int_0^1 \Bigl(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dx\Bigr)\,dy = \int_0^1 \frac{-1}{1+y^2}\,dy = -\tan^{-1}(y)\Big|_{y=0}^{y=1} = -\frac{\pi}{4}.$$

We know that

$$\int_0^1 \Bigl(\int_0^1 \Bigl|\frac{x^2 - y^2}{(x^2+y^2)^2}\Bigr|\,dy\Bigr)\,dx = \infty,$$

because otherwise Fubini's Theorem would be contradicted. But let us verify this directly:

$$\begin{aligned}
\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy &= \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy + \int_x^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dy \\
&= \int_0^x \frac{\partial}{\partial y}\Big(\frac{y}{x^2+y^2}\Big)\,dy + \int_x^1 \frac{\partial}{\partial y}\Big(\frac{-y}{x^2+y^2}\Big)\,dy \\
&= \frac{y}{x^2+y^2}\Big|_{y=0}^{y=x} + \frac{-y}{x^2+y^2}\Big|_{y=x}^{y=1} \\
&= \frac{1}{2x} + \Big(\frac{1}{2x} - \frac{1}{x^2+1}\Big) \\
&= \frac{1}{x} - \frac{1}{1+x^2}.
\end{aligned}$$

Clearly

$$\int_0^1 \frac{1}{x}\,dx = \infty \quad \text{and} \quad \int_0^1 \frac{1}{1+x^2}\,dx < \infty,$$

and so

$$\int_0^1 \Big(\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy\Big)\,dx = \infty.$$

2. For $t > 0$ consider $f(t) := \Big(\int_0^t e^{-x^2}\,dx\Big)^2 + \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$

(a) Show that f is constant.

Solution: We show that f is differentiable with zero derivative. By the fundamental theorem of calculus

$$\frac{d}{dt}\int_0^t e^{-x^2}\,dx e^{-t^2},$$

so by the chain rule and the substitution $y = x/t$

$$\frac{d}{dt}\Big(\int_0^t e^{-x^2}\,dx\Big)^2 = 2e^{-t^2}\int_0^t e^{-x^2}\,dx = 2te^{-t^2}\int_0^1 e^{-t^2y^2}\,dy = 2t\int_0^1 e^{-t^2(1+x^2)}\,dx.$$

The other integral is a parameter integral. Assuming that we can differentiate under the integral we get

$$\frac{d}{dt}\int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx = \int_0^1 \frac{\partial}{\partial t}\frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx = -2t\int_0^1 e^{-t^2(1+x^2)}\,dx$$

and therefore

$$f'(t) = 2t\int_0^1 e^{-t^2(1+x^2)}\,dx - 2t\int_0^1 e^{-t^2(1+x^2)}\,dx = 0$$

for all $t > 0$. We finally need to check whether we can differentiate under the integral sign. We note that

$$\left|\frac{\partial}{\partial t}\frac{e^{-t^2(1+x^2)}}{1+x^2}\right| = 2te^{-t^2(1+x^2)} \leq 2T$$

for all $t \in [0, T]$. Since $\int_0^1 2T\,dx = 2T < \infty$ the theorem on the differentiation of parameter integrals applies, showing that the integral is differentiable for $t \in (0, T)$. Since this works for all $T > 0$ we conclude that f is differentiable for all $t > 0$.

(b) By looking at $f(t)$ as $t \to 0$ and $t \to \infty$ deduce that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.

Solution: Since f is constant we have

$$\lim_{t\to 0} f(t) = \lim_{t\to\infty} f(t).$$

We compute the two limits. Clearly

$$\int_0^t e^{-x^2}\,dx \to \int_0^0 e^{-x^2}\,dx = 0$$

as $t \to 0$ and

$$\int_0^t e^{-x^2}\,dx \to \int_0^\infty e^{-x^2}\,dx$$

as $t \to \infty$. Now

$$\Big|\frac{e^{-t^2(1+x^2)}}{1+x^2}\Big| \leq \frac{1}{1+x^2}$$

for all $x \in [0,1]$ and all $t \geq 0$. Since the right hand side has a finite integral over $[0,1]$ we can apply the theorem on the continuity of parameter integrals and conclude that

$$\int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx \to \int_0^1 \frac{0}{1+x^2}\,dx = 0$$

as $t \to 0$ and

$$\int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4}$$

as $t \to \infty$. Combining everything we get

$$\frac{\pi}{2}\lim_{t\to 0} f(t) = \lim_{t\to\infty} f(t) = \Big(\int_0^\infty e^{-x^2}\,dx\Big)^2.$$

Taking square roots on both sides our claim follows.

(c) Use (b) and Fubini's Theorem to show that $\displaystyle\int_{\mathbb{R}^N} e^{-|x|^2}\,dx = \pi^{N/2}$, where $|x|$ is the Euclidean norm of x.

Solution: We give a proof by induction. For $N = 1$ we have

$$\int_{\mathbb{R}} e^{-x^2}\,dx = 2\int_0^\infty e^{-x^2}\,dx = 2\frac{\sqrt{\pi}}{2} = \pi^{1/2}.$$

Suppose now that the assertion is true for $N-1$, that is,

$$\int_{\mathbb{R}^{N-1}} e^{-|x|^2}\,dx = \pi^{(N-1)/2}.$$

If $x \in \mathbb{R}^N$, then we can write $x = (y, z)$ with $y \in \mathbb{R}^{N-1}$ and $z \in \mathbb{R}$. Then $|x|^2 = |y|^2 + z^2$ by definition of the Euclidean norm. Hence by Fubini's (or more specifically Tonelli's theorem) we have

$$\begin{aligned}\int_{\mathbb{R}^N} e^{-|x|^2}\,dx &= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} e^{-|y|^2-z^2}\,dz\,dy\\ &= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} e^{-|y|^2}e^{-z^2}\,dz\,dy = \Big(\int_{\mathbb{R}^{N-1}} e^{-|y|^2}\,dy\Big)\Big(\int_{\mathbb{R}} e^{-z^2}\,dz\Big).\end{aligned}$$

Using the induction assumption and the case $N = 1$ we get

$$\int_{\mathbb{R}^N} e^{-|x|^2}\,dx = \pi^{(N-1)/2}\pi^{1/2} = \pi^{N/2}$$

as claimed.

3. Let $\mathcal{A}$ be a σ-algebra of subsets of X and $\mathcal{B}$ a σ-algebra of subsets of Y. Denote by $\mathcal{A}\times\mathcal{B}$ the smallest σ-algebra of subsets of $X \times Y$ such that it contains all generalised rectangles $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $C \in \mathcal{A}\times\mathcal{B}$. For $x \in X$ set $C_x := \{y \in Y : (x,y) \in C\}$ and for $y \in Y$ set $C^y := \{x \in X : (x,y) \in C\}$. Define $\mathcal{S} := \{C \in \mathcal{A}\times\mathcal{B} : C_x \in \mathcal{B} \text{ for all } x \in X\}$

(a) Show that every measurable generalised rectangle in $\mathcal{A} \times \mathcal{B}$ is in $\mathcal{S}$.

Solution: Let $C = A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then

$$C_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Since B and $\emptyset$ are measurable $C \in \mathcal{S}$.

(b) Show that $\mathcal{S}$ is a σ-algebra.

Solution: We prove the three properties of a σ-algebra.
(i) Clearly $\emptyset = \emptyset \times \emptyset \in \mathcal{S}$.
(ii) Suppose now that $C \in \mathcal{S}$ and fix $x \in X$. If $y \in C_x^c$, then $(x, y) \notin C$ and so $(x, y) \in C^c$. Hence $y \in (C^c)_x$. On the other hand, if $y \in (C^c)_x$, then $(x, y) \in C^c$ and so $(x, y) \notin C$. Hence $y \notin C_x$, that is, $y \in C_x^c$. Therefore $(C^c)_x = C_x^c \in \mathcal{B}$ since $\mathcal{B}$ is a σ-algebra.
(iii) Suppose now that $C_k \in \mathcal{S}$ for all $k \in \mathbb{N}$. If $y \in (\bigcup_{k\in\mathbb{N}} C_k)_x$, then $(x, y) \in \bigcup_{k\in\mathbb{N}} C_k$. Hence there exists k_0 with $(x, y) \in C_{k_0}$, and so $y \in (C_{k_0})_x$. This shows that $(\bigcup_{k\in\mathbb{N}} C_k)_x \subseteq \bigcup_{k\in\mathbb{N}}(C_k)_x$. If $y \in \bigcup_{k\in\mathbb{N}}(C_k)_x$, then there exists k_0 such that $y \in (C_{k_0})_x$. But then $(x, y) \in C_{k_0}$ and so $(x, y) \in \bigcup_{k\in\mathbb{N}} C_k$. Hence $\bigcup_{k\in\mathbb{N}}(C_k)_x \subseteq (\bigcup_{k\in\mathbb{N}} C_k)_x$. By putting together both we get

$$\bigcup_{k\in\mathbb{N}}(C_k)_x = (\bigcup_{k\in\mathbb{N}} C_k)_x \in \mathcal{B}$$

since $(C_k)_x \in \mathcal{B}$ for all $k \in \mathbb{N}$ and $\mathcal{B}$ is a σ-algebra.

(c) Show that $C_x \in \mathcal{B}$ and $C^y \in \mathcal{A}$ whenever $C \in \mathcal{A} \times \mathcal{B}$.

Solution: Since $\mathcal{S}$ is a σ-algebra containing all generalised measurable rectangles and $\mathcal{A} \times \mathcal{B}$ is the σ-algebra generated by the measurable rectangles we have $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{S}$. Interchanging the roles of x and y also $\mathcal{S}' := \{C \in \mathcal{A} \times \mathcal{B} \colon C^y \in \mathcal{A}\}$ is a σ-algebra a similar argument shows that $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{S}'$. Hence

$$\mathcal{A} \times \mathcal{B} \subseteq \mathcal{S} \cap \mathcal{S}'.$$

This implies that $C_x \in \mathcal{B}$ and $C^y \in \mathcal{A}$ whenever $C \in \mathcal{A} \times \mathcal{B}$.

Extra questions for further practice

4. (a) Use Fubini's theorem and the relation $\frac{1}{x} = \int_0^\infty e^{-xy}\, dy$ to show that

$$\lim_{a\to\infty} \int_0^a \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Solution: by using the expression for $1/x$ above we have

$$\int_0^a \frac{\sin x}{x}\, dx = \int_0^a \int_0^\infty e^{-xy} \sin x\, dx\, dy.$$

To be able to interchange the order of integration we check the conditions of Fubini's theorem. Using that $|\sin x| \le x$ for all $x \ge 0$ we get

$$\int_0^a \int_0^\infty |e^{-xy} \sin x|\, dx\, dy \le \int_0^a \int_0^\infty e^{-xy} x\, dx\, dy = \int_0^a \frac{x}{x}\, dy = a$$

for all $a > 0$. Hence we can interchange the integrals and get

$$\int_0^a \frac{\sin x}{x}\, dx = \int_0^\infty \int_0^a e^{-xy} \sin x\, dy\, dx.$$

To compute the inner integral note that $\sin x = \operatorname{Im} e^{ix}$. Now

$$\int_0^a e^{-xy} e^{ix}\, dx = \int_0^a e^{x(i-y)}\, dy = \frac{1}{i-y} e^{x(i-y)}\Big|_0^a = \frac{y+i}{1+y^2}\big(1 - e^{ia}e^{-ay}\big).$$

Hence

$$\int_0^a e^{-xy} \sin x\, dy = \operatorname{Im} \int_0^a e^{-xy} e^{ix}\, dy = \operatorname{Im}\Big(\frac{y+i}{1+y^2}\big(1 - e^{ia}e^{-ax}\big)\Big)$$
$$= \frac{1}{1+y^2} - \frac{e^{-ay}}{1+y^2}\big(\cos ax + y \sin ax\big).$$

Alternatively we could compute the above integral by integrating by parts twice. Now clearly

$$\frac{1}{1+y^2} - \frac{e^{-ay}}{1+y^2}\big(\cos ax + y \sin ax\big) \to 0$$

as $a \to \infty$ for all $y > 0$. Since $ye^{-ay} \leq 1$ for all $a \geq 1$ we have

$$\Big|\frac{e^{-ay}}{1+y^2}\big(\cos ax + y \sin ax\big)\Big| \leq \frac{2}{1+y^2}$$

for all $y \geq 0$ and all $a \geq 1$. Since $1/(1+y^2)$ is integrable we can apply the theorem on the continuity of parameter integrals to conclude that

$$\lim_{a\to\infty} \int_0^a \frac{\sin x}{x}\, dx = \lim_{a\to\infty} \int_0^\infty \Big(\frac{1}{1+y^2} - \frac{e^{-ay}}{1+y^2}\big(\cos ax + y \sin ax\big)\Big)\, dy$$
$$= \int_0^\infty \frac{1}{1+y^2}\, dy = \tan^{-1} y\Big|_0^\infty = \frac{\pi}{2}$$

as claimed.

(b) Prove that $\dfrac{\sin x}{x} \notin \mathcal{L}^1((0,\infty),\mathbb{R})$.

Solution: We can write

$$\int_0^\infty \frac{|\sin x|}{x}\, dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\, dx$$
$$\geq \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{(n+1)\pi}\, dx = \frac{1}{\pi} \int_0^\pi \sin x\, dx \sum_{n=1}^\infty \frac{1}{n}.$$

Since the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges we have

$$\int_0^\infty \frac{|\sin x|}{x}\, dx = \infty$$

as claimed.

5. Show that

$$\int_0^\infty e^{-y} \frac{\sin^2(y)}{y}\, dy = \frac{\log 5}{4}.$$

by applying Fubini's theorem, looking at the integrand as an integral of $f(x,y) = \sin(2xy)e^{-y}$.

Solution: Consider

$$\int_0^\infty \Big(\int_0^1 \sin(2xy)e^{-y}\, dx\Big)\, dy.$$

The inner integral equals

$$e^{-y}\int_0^1 \sin(2xy)\,dx = e^{-y}\frac{-\cos(2xy)}{2y}\Big|_{x=0}^{x=1} = e^{-y}\frac{1-\cos(2y)}{2y} = e^{-y}\frac{\sin^2(y)}{y}.$$

Let $I(x) = \int_0^\infty \sin(2xy)e^{-y}\,dy$. Two integrations by parts yield

$$I(x) = 2x - 4x^2\int_0^\infty \sin(2xy)e^{-y}\,dy,$$

so that $I(x) = 2x/(1+4x^2)$. Thus

$$\int_0^1\Big(\int_0^\infty \sin(2xy)e^{-y}\,dy\Big)\,dx = \int_0^1 \frac{2x}{1+4x^2}\,dx = \frac{1}{4}\log(1+4x^2)\Big|_{x=0}^{x=1} = \frac{\log 5}{4}.$$

Fubini's Theorem is applicable because

$$\int_0^\infty\Big(\int_0^1 \big|\sin(2xy)e^{-y}\big|\,dx\Big)\,dy \le \int_0^\infty\Big(\int_0^1 e^{-y}\,dx\Big)\,dy = \int_0^\infty e^{-y}\,dy = 1 < \infty.$$

Thus

$$\int_0^\infty e^{-y}\frac{\sin^2(y)}{y}\,dy = \frac{\log 5}{4}.$$

Challenge questions (optional)

6. By a complete measure μ we mean a measure such that if M is measurable and $\mu(M) = 0$, then every subset of M is measurable (and has measure zero). The Lebesgue measure defined on the Lebesgue σ-algebra is complete. It is not complete when only defined on the Borel σ-algebra.

Suppose that μ is a complete measure defined on the σ-algebra $\mathcal{A}$ of subsets of X. Let $f\colon X \to \mathbb{R}$ be a measurable function and $g\colon X \to \mathbb{R}$ a function with $f = g$ almost everywhere. Prove that g is measurable. Is this still true if μ is not complete?

Solution: By assumption there exists a measurable set $N \subseteq X$ with $\mu(X) = 0$ such that $f(x) = g(x)$ for all $x \in X \setminus N$. Mow consider the set

$$N_0 := \{x \in X\colon f(x) \neq g(x)\}$$

Then clearly $N_0 \subseteq N$ and by the completeness of μ the set N_0 is measurable. Now let $U \subseteq \mathbb{R}$ be open. If $0 \notin U$, then

$$\{x \in X\colon g(x) - f(x) \in U\} \subseteq N_0$$

is measurable because of the completeness of μ. Similarly

$$\{x \in X\colon g(x) - f(x) = 0\} = X \setminus N_0$$

is measurable. Hence if U is open and $0 \in U$, then $U \setminus \{0\}$ is open and so

$$\{x \in X\colon g(x) - f(x) \in U\} = \{x \in X\colon g(x) - f(x) \in U \setminus \{0\}\} \cup \{x \in X\colon g(x) - f(x) = 0\}$$

is measurable. Hence $g - f$ is a measurable function. Hence $g = g + (f - g)$ is measurable because it is the sum of two measurable functions.

Assume now that μ is not complete. Then there exist sets N_0, N such that $N_0 \subset N$, N is measurable, but N_0 is not measurable. Then the constant function $f = 0$ is measurable and $g := 1_{N_0}$ is not measurable. However, $f = g$ almost everywhere. Hence if f is measurable and $f = g$ almost everywhere, then g does not need to be measurable.