The University of Sydney
School of Mathematics and Statistics

# Tutorial 8 (Week 9)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Questions marked with * are harder questions.

## Material covered

(1) Convolution and applications

(2) Applications of Hölder's inequality

(3) Properties of convex functions

(4) Applications of parameter integrals and dominated convergence theorem

## Outcomes

After completing this tutorial you should

(1) be able to compute simple convolution integrals

(2) be able to prove some properties of convolution

(3) be aware of the issues arising for convolution with singular functions.

(4) be able to prove inequalities for convex functions, and deriving from Hölder's inequality.

## Questions to complete during the tutorial

**1.** Let $f := 1_{[0,1]}$ be the indicator function of $[0, 1]$ on $\mathbb{R}$. Calculate the convolution $f * f$ and sketch its graph.

**2.** Let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$ with $1/\infty := 0$ by convention. Show that convolution $*\colon L^p(\mathbb{R}^N, \mathbb{C}) \times L^q(\mathbb{R}^N, \mathbb{C}) \to BC(\mathbb{R}^N, \mathbb{C})$, where $BC(\mathbb{R}^N, \mathbb{C})$ is the space of bounded and continuous functions from $\mathbb{R}^N$ to $\mathbb{C}$.

*Hint:* Use the definition of convolution and the continuity of translations on $L^q(\mathbb{R}^N, \mathbb{C})$ for $1 \leq q < \infty$.

**3.** Let $N(x) := \dfrac{1}{|x|}$ denote the Newtonian potential of a unit point mass located at the origin. The potential of a body of mass density $\varrho(x)$ is given by the convolution

$$V(x) = (N * \varrho)(x) = \int_{\mathbb{R}^3} \frac{\varrho(y)}{|x - y|}\, dy$$

(Convolution is the superposition of the potential of masses $\rho(y)$ over the whole space.)

(a) If $\varrho \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^N)$, show that $V \in BC(\mathbb{R}^3)$.

(b) Show that $\Delta N(x) = 0$ for all $x \neq 0$. ($\Delta u := \operatorname{div}(\operatorname{grad} u)$ is the *Laplace operator*)

*(c) Assume that $\varrho \in L^\infty(\mathbb{R}^3)$ with bounded support. Show that $\Delta V(x) = 0$ if $x \notin \operatorname{supp}(\varrho)$.

*(d) If $\varrho \in C_c^2(\mathbb{R}^3)$, show that $-\Delta V = 4\pi\varrho$.
*Hint:* Cut out a ball about the singularity and apply Green's formula.

 

## Extra questions for further practice

**4.** Let $f \in L_p(X,\mathbb{C}) \cap L_q(X,\mathbb{C})$ with $1 \le p \le q \le \infty$. Use Hölder's inequality to prove that

$$\|f\|_r \le \|f\|_p^\theta \|f\|_q^{1-\theta}$$

if $p \le r \le q$ and $1/r = \theta/p + (1-\theta)/q$.

**5.** A measure space $(X, \mathcal{A}, \mu)$ is called $\sigma$-finite if there exists a sequence $(X_n)$ of subsets with $X = \bigcup_{n\in\mathbb{N}} X_n$ and $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. Let $\mu$ be a $\sigma$-finite measure defined on the set $X$.

(a) Show that there exist measurable sets $B_n \subseteq X$ with $\mu(B_n) < \infty$, $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ and $X = \bigcup_{n\in\mathbb{N}} B_n$.

(b) Show that there exist measurable sets $A_n \subseteq X$ with $\mu(A_n) < \infty$, $A_n \cap A_k = \emptyset$ if $n \ne k$ and $X = \bigcup_{n\in\mathbb{N}} A_n$.

**6.** Let $\mu$ be a measure defined on $X$. Show that $\mu$ is $\sigma$-finite if and only if there exists a measurable function $f\colon X \to (0,\infty)$ with $\int_X f\,d\mu < \infty$.

**7.** A function $\varphi\colon (a,b) \to \mathbb{R}$ is called convex if $\varphi(\lambda x + (1-\lambda)y) \le \lambda\varphi(x) + (1-\lambda)\varphi(y)$ for all $x, y \in (a,b)$ and all $\lambda \in [0,1]$.

(a) Show that $\varphi\colon (a,b) \to \mathbb{R}$ is convex if and only if

$$\frac{\varphi(t)-\varphi(s)}{t-s} \le \frac{\varphi(u)-\varphi(t)}{u-t} \tag{1}$$

whenever $a < s < t < u < b$.
*Hint:* Choose $\lambda \in (0,1)$ such that $t = (1-\lambda)s + \lambda u$.

(b) Let $\varphi\colon (a,b) \to \mathbb{R}$ be convex. Use (1) to show that

$$\frac{\varphi(t)-\varphi(s)}{t-s} \le \frac{\varphi(u)-\varphi(v)}{u-v} \tag{2}$$

whenever $a < s < t < u < v < b$.

(c) Show that a differentiable function $\varphi\colon (a,b) \to \mathbb{R}$ is convex if and only if $\varphi'$ is increasing.

## Challenge questions (optional)

**8.** Define

$$L^1_{\text{loc}}(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ measurable}, \int_B |f|\,dx < \infty \text{ for every } B \subseteq \mathbb{R} \text{ bounded} \right\}.$$

Let $f \in L^1_{\text{loc}}(\mathbb{R})$. We say that $f$ is *weakly differentiable* if there exists $g \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} g\varphi\,dx = -\int_{\mathbb{R}} f\varphi'\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R})$.

(a) If it exists, show that the weak derivative of a function is unique almost everywhere.

(b) If $f$ is differentiable, show that $g = f'$ is the weak derivative of $f$.

(c) Compute the weak derivative of $f(x) = |x|$.

(d) Show that $H(x) := 1_{[0,\infty)}$ is not weakly differentiable.