The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 8 (Week 9)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Questions marked with * are harder questions.

Material covered

(1) Convolution and applications

(2) Applications of Hölder's inequality

(3) Properties of convex functions

(4) Applications of parameter integrals and dominated convergence theorem

Outcomes

After completing this tutorial you should

(1) be able to compute simple convolution integrals

(2) be able to prove some properties of convolution

(3) be aware of the issues arising for convolution with singular functions.

(4) be able to prove inequalities for convex functions, and deriving from Hölder's inequality.

Questions to complete during the tutorial

1. Let $f := 1_{[0,1]}$ be the indicator function of $[0,1]$ on $\mathbb{R}$. Calculate the convolution $f*f$ and sketch its graph.

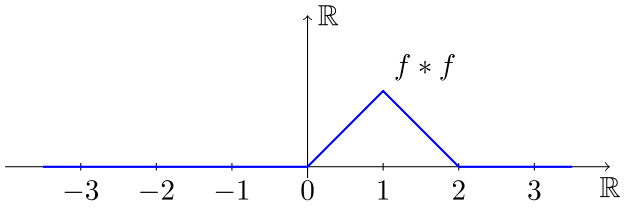
Solution: For fixed $x \in \mathbb{R}$, $x-y \in [0,1]$ holds if and only if $y \in [x-1,x]$. Hence

$$\begin{aligned}(f*f)(x) &= \int_{-\infty}^{\infty} f(x-y)f(y)\,dy = \int_{-\infty}^{\infty} 1_{[0,1]}(x-y)1_{[0,1]}(y)\,dy \\ &= \int_{-\infty}^{\infty} 1_{[x-1,x]}(y)1_{[0,1]}(y)\,dy = \int_{-\infty}^{\infty} 1_{[x-1,x]\cap[0,1]}(y)\,dy = m\big([x-1,x]\cap[0,1]\big).\end{aligned}$$

Consider the intersection $[x-1,x]\cap[0,1]$ as x varies. If $x<0$, the intersection is empty. If $0 \leq x \leq 1$, it is $[0,x]$, with measure x. If $1 \leq x \leq 2$ it is $[x-1,1]$, with measure $2-x$. If $x>2$, the intersection is again empty. Hence

$$(f*f)(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 2-x & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

and $f*f$ has graph:



Notice that $f * f$ is continuous, even though f is not continuous. In Question 2 we see a general reason for this.

2. Let $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$ with $1/\infty := 0$ by convention. Show that convolution $*\colon L^p(\mathbb{R}^N, \mathbb{C}) \times L^q(\mathbb{R}^N, \mathbb{C}) \to BC(\mathbb{R}^N, \mathbb{C})$, where $BC(\mathbb{R}^N, \mathbb{C})$ is the space of bounded and continuous functions from $\mathbb{R}^N$ to $\mathbb{C}$.

Hint: Use the definition of convolution and the continuity of translations on $L^q(\mathbb{R}^N, \mathbb{C})$ for $1 \leq q < \infty$.

Solution: By definition of convolution, Hölder's inequality, and the translation invariance of integrals

$$\big|(f * g)(x)\big| \leq \int_{\mathbb{R}^N} |f(z)||g(x-z)|\,dz \leq \|f\|_p \|\tau_x g\|_q = \|f\|_p \|g\|_q$$

for all $x \in \mathbb{R}^N$, where $\tau_x(z) = g(x - z)$ is translation. Hence f is bounded and $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. We next show the continuity of $f * g$. We assume that $1 \leq q < \infty$. By definition of convolution

$$\begin{aligned}\big|(f * g)(x) - (f * g)(y)\big| &= \Big|\int_{\mathbb{R}^N} f(z) g(x-z)\,dz - \int_{\mathbb{R}^N} f(z) g(z-y)\,dz\Big| \\ &= \Big|\int_{\mathbb{R}^N} f(z)\big(g(x-z) - g(y-z)\big)\,dz\Big| \leq \int_{\mathbb{R}^N} |f(z)|\big|g(x-z) - g(y-z)\big|\,dz.\end{aligned}$$

Applying Hölder's inequality and the translation invariance of the integral we get

$$\big|(f * g)(x) - (f * g)(y)\big| \leq \int_{\mathbb{R}^N} |f(z)|\big|g(x-z) - g(y-z)\big|\,dz \leq \|f\|_p \|\tau_{x-y}\check{g} - \check{g}\|_q,$$

where $\check{g}(x) := g(-x)$ is the reflection of x. By continuity of translation on $L^q(\mathbb{R}^N, \mathbb{C})$ we get

$$\big|(f * g)(x) - (f * g)(y)\big| \leq \|f\|_p \|\tau_{x-y}\check{g} - \check{g}\|_q \to 0$$

as $y \to x$. Hence $f * g$ is continuous. If $q = \infty$ we use the fact that $f * g = g * f$ and use what we proved above.

3. Let $N(x) := \dfrac{1}{|x|}$ denote the Newtonian potential of a unit point mass located at the origin. The potential of a body of mass density $\varrho(x)$ is given by the convolution

$$V(x) = (N * \varrho)(x) = \int_{\mathbb{R}^3} \frac{\varrho(y)}{|x-y|}\,dy$$

(Convolution is the superposition of the potential of masses $\rho(y)$ over the whole space.)

(a) If $\varrho \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^N)$, show that $V \in BC(\mathbb{R}^3)$.

Solution: Integration in spherical coordinates shows that

$$\int_{|x|\leq 1} N(x)\,dx = 4\pi \int_0^1 \frac{1}{r} r^2\,dr = 2\pi r^2 < \infty.$$

and $N(x) \leq 1$ for $|x| \geq 1$. Hence, if $B := B(0,1)$ is the unit ball, then $N_1 := 1_B N \in L^1(\mathbb{R}^N)$, $N_2 := 1_{B^c} N$ and $N = N_1 + N_2$. Hence by Question 2 we have

$$N * \varrho = N_1 * \varrho + N_2 * \varrho \in BC(\mathbb{R}^3)$$

since each convolution is a pairing of an L^1 and an L^∞ function, using that ϱ is in both by assumption.

(b) Show that $\Delta N(x) = 0$ for all $x \neq 0$. ($\Delta u := \operatorname{div}(\operatorname{grad} u)$ is the *Laplace operator*)

Solution: Note that

$$\frac{\partial}{\partial x_i}|x| = \frac{\partial}{\partial x_i}\sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{2x_i}{2\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{x_i}{|x|}$$

for $i = 1, 2, 3$. Hence by the chain rule

$$\operatorname{grad}\frac{1}{|x|} = -\frac{x}{|x|^3}$$

and therefore by the chain and product rules

$$\Delta\frac{1}{|x|} = -\operatorname{div}\frac{x}{|x|^3} = -\frac{3}{|x|^3} + 3\frac{x \cdot x}{|x|^5} = -\frac{3}{|x|^3} + \frac{3}{|x|^3} = 0$$

for all $x \neq 0$.

*(c) Assume that $\varrho \in L^\infty(\mathbb{R}^3)$ with bounded support. Show that $\Delta V(x) = 0$ if $x \notin \operatorname{supp}(\varrho)$.

Solution: Fix $x_0 \in \mathbb{R}^3$ such that $x_0 \notin \operatorname{supp}(\varrho)$. We can then choose $\varepsilon > 0$ such that $B(x_0, 2\varepsilon) \cap \operatorname{supp}(\varrho) = \emptyset$. Also let $R > 0$ such that $\operatorname{supp}(\varrho) \subseteq B(0,R)$. Now note that if $x \in B(x_0, \varepsilon)$ and $|x - y| < \varepsilon$, then $|x_0 - y| \leq |x_0 - x| + |x - y| < 2\varepsilon$. As $\varrho(y) = 0$ for $y \in B(x_0, 2\varepsilon)$ we have that

$$\Big|\frac{\partial}{\partial x_i}\frac{\varrho(y)}{|x-y|}\Big| = \frac{|x_i - y_i||\varrho(y)|}{|x-y|^3} \leq \frac{2\varepsilon}{\varepsilon^3}|\varrho(y)| = \frac{2}{\varepsilon^2}|\varrho(y)|.$$

Since ϱ has compact support it is integrable, and so the theorem on the differentiation of parameter integral tells us that

$$\frac{\partial}{\partial x_i}V(x) = \int_{\mathbb{R}^3}\frac{(x_i - y_i)\varrho(y)}{|x-y|^3}\,dy$$

for all $x \in B(x_0, \varepsilon)$. Similarly,

$$\Big|\frac{\partial^2}{\partial x_i^2}\frac{\varrho(y)}{|x-y|}\Big| = \frac{|\varrho(y)|}{|x-y|^3} + \frac{(x_i - y_i)^2|\varrho(y)|}{|x-y|^5} \leq \Big(\frac{1}{\varepsilon^3} + \frac{4}{\varepsilon^3}\Big)|\varrho(y)|$$

for all $x \in B(x_0, \varepsilon)$. Hence the theorem on the differentiation of parameter integrals shows that

$$\Delta V(x) = \int_{\mathbb{R}^3}\varrho\Delta\frac{1}{|x-y|}\,dy = \int 0\varrho(y)\,dy = 0.$$

for all $x \in B(x_0, \varepsilon)$.

*(d) If $\varrho \in C_c^2(\mathbb{R}^3)$, show that $-\Delta V = 4\pi\varrho$.

Hint: Cut out a ball about the singularity and apply Green's formula.

Solution: If we naïvely interchange differentiation and integration as in the previous part, we get zero, but that cannot be correct. The argument in the previous part relied on the fact that we could "blend out" the singularity of the potential, by using that $\varrho(y) = 0$ in a neighbourhood of the singularity. In general we cannot do this, so we need another way to compute the derivatives of V. The integral $N * \varrho$ is a *singular integral* whose properties are much more difficult to derive. This is one of the main aims of *harmonic analysis*. To solve the question we use that

$$(N * \varrho)(x) = (\varrho * N)(x) = \int_{\mathbb{R}^3} \frac{\varrho(x-y)}{|y|}\, dy$$

As $\varrho \in C_c^2(\mathbb{R}^3)$ we have

$$\frac{\partial \varrho}{\partial x_i}, \frac{\partial^2 \varrho}{\partial x_i^2} \in C_c(\mathbb{R}^3).$$

Let now $\operatorname{supp}(\varrho) \subseteq B(0,R)$ and

$$M := \max\Big\{\Big\|\frac{\partial \varrho}{\partial x_i}\Big\|_\infty, \Big\|\frac{\partial^2 \varrho}{\partial x_i^2}\Big\|_\infty\Big\}$$

If we fix $x_0 \in \mathbb{R}^3$ and $\delta > 0$, then

$$\Big|\frac{\partial}{\partial x_i}\frac{\varrho(x-y)}{|y|}\Big|, \Big|\frac{\partial^2}{\partial x_i^2}\frac{\varrho(x-y)}{|y|}\Big| \leq \frac{M}{|y|}1_{B(0,R+\varepsilon)}$$

for all $x \in B(x_0,\delta)$ and all $y \in \mathbb{R}^N$. Hence the theorem on the differentiation of parameter integrals applies and so

$$\Delta V = \int_{\mathbb{R}^3} \frac{\Delta\varrho(x-y)}{|y|}\, dy = \int_{\mathbb{R}^3} \frac{\Delta\varrho(y)}{|x-y|}\, dy$$

for all $x \in B(x_0,\varepsilon)$. Since the above applies to every choice of $x_0 \in \mathbb{R}^3$ the formula holds for all $x \in \mathbb{R}^3$. We next use Green's formula

$$\int_\Omega u\Delta v - v\Delta u\, dy = \int_{\partial\Omega} (u\nabla v - v\nabla u)\cdot \nu\, dS,$$

where ν is the outer unit normal to the region Ω and dS integration with respect to surface measure. (This is a consequence of the divergence theorem, see vector calculus). Green's formula allows us to to move the Laplacian from ϱ to the Newtonian potential. We do that for fixed $x \in \mathbb{R}^3$. We have to choose the region first. We note that

$$\int_{\mathbb{R}^3} \frac{\Delta\varrho(y)}{|x-y|}\, dy = \lim_{\delta\to 0^+} \int_{\delta<|x-y|<R_0} \frac{\Delta\varrho(y)}{|x-y|}\, dy,$$

so the region we choose is the annulus $\delta < |x-y| < R_0$. where we choose R_0 large enough so that $\varrho = 0$ and also $\nabla\varrho = 0$. This is possible since ϱ has compact support. Hence by Green's formula and since the outward unit normal on $|x-y| = \delta$ is $(x-y)/\delta$

$$\begin{aligned}\int_{\delta<|x-y|<R_0} \frac{\Delta\varrho(y)}{|x-y|}\, dy &= \int_{\delta<|x-y|<R_0} \varrho(y)\Delta\frac{1}{|x-y|}\, dy \\ &\quad + \int_{|x-y|=\delta} \frac{(x-y)\cdot\nabla\varrho(y)}{\delta|x-y|}\, dS - \int_{|x-y|=\delta} \varrho(y)\frac{(x-y)\cdot(x-y)}{\delta|x-y|^3}\, dS\end{aligned}$$

The first integral on the right hand side is zero since $\Delta\dfrac{1}{x-y} = 0$ for $|x-y| > \delta$. For the second integral we use that the surface area of a sphere of radius δ is $4\pi\delta^2$, and so

$$\Big|\int_{|x-y|=\delta} \frac{(x-y)\cdot\nabla\varrho(y)}{\delta|x-y|}\, dS\Big| \leq \int_{|x-y|=\delta} \frac{|\nabla\varrho(y)|}{\delta}\, dS \leq \frac{4\pi\delta^2}{\delta}\|\nabla\varrho\|_\infty \to 0$$

as $\delta \to 0$. The last integral is equal to

$$\frac{1}{\delta^2}\int_{|x-y|=\delta} \varrho(y)\, dS$$

We would like to show that the above converges to $4\pi\varrho(x)$. To do so note that

$$\begin{aligned}\Big|\frac{1}{\delta^2}\int_{|x-y|=\delta}\varrho(y)\,dS-4\pi\varrho(x)\Big| &= \frac{1}{\delta^2}\Big|\int_{|x-y|=\delta}\varrho(y)-\varrho(x)\,dS\Big| \\ &\leq \frac{1}{\delta^2}\int_{|x-y|=\delta}|\varrho(y)-\varrho(x)|\,dS.\end{aligned}$$

Now fix $\varepsilon>0$. By the continuity of ϱ there exists $\delta_0>0$ such that $|\varrho(y)-\varrho(x)|<\varepsilon$ if $|x-y|<\delta_0$. Therefore

$$\int_{|x-y|=\delta}|\varrho(y)-\varrho(x)|\,dS \leq \frac{\varepsilon}{\delta^2}\int_{|x-y|=\delta}1\,dS = 4\pi\varepsilon$$

for all $0<\delta<\delta_0$. As the above argument works for every choice of $\varepsilon>0$ we get

$$-\int_{|x-y|=\delta}\varrho(y)\frac{(x-y)\cdot(x-y)}{\delta|x-y|^3}\,dS \to 4\pi\varrho(x)$$

as $\delta\to 0$. This concludes the proof.

Extra questions for further practice

4. Let $f\in L_p(X,\mathbb{C})\cap L_q(X,\mathbb{C})$ with $1\leq p\leq q\leq\infty$. Use Hölder's inequality to prove that

$$\|f\|_r \leq \|f\|_p^\theta\|f\|_q^{1-\theta}$$

if $p\leq r\leq q$ and $1/r=\theta/p+(1-\theta)/q$.

Solution: First assume that $1\leq p<r<q=\infty$. Then

$$\Big(\int_X|f|^r\,d\mu\Big)^{1/r} = \Big(\int_X|f|^p|f|^{r-p}\,d\mu\Big)^{1/r} \leq \Big(\|f\|_\infty^{r-p}\int_X|f|^p\,d\mu\Big)^{1/r} = \|f\|_p^{p/r}\|f\|_\infty^{1-p/r}.$$

Hence $\theta=p/r$ so that $1/r=\theta/p+(1-\theta)/\infty$. If $1\leq p<r<q<\infty$ we use the generalised Hölder inequality from Tutorials. From the assumptions we have

$$\frac{1}{r}=\frac{1}{\frac{p}{\theta}}+\frac{1}{\frac{q}{1-\theta}}.$$

Hence by the definition of the L^p-norms

$$\|f\|_r = \||f|^\theta|f|^{1-\theta}\|_r \leq \||f|^\theta\|_{\frac{p}{\theta}}\||f|^{1-\theta}\|_{\frac{q}{1-\theta}} = \|f\|_p^\theta\|f\|_q^{1-\theta}$$

as claimed.

5. A measure space $(X,\mathcal{A},\mu)$ is called σ-finite if there exists a sequence (X_n) of subsets with $X=\bigcup_{n\in\mathbb{N}}X_n$ and $\mu(X_n)<\infty$ for all $n\in\mathbb{N}$. Let μ be a σ-finite measure defined on the set X.

(a) Show that there exist measurable sets $B_n\subseteq X$ with $\mu(B_n)<\infty$, $B_0\subseteq B_1\subseteq B_2\subseteq\dots$ and $X=\bigcup_{n\in\mathbb{N}}B_n$.

Solution: By definition of a σ-finite measure there exist $X_k\subseteq X$ such that $\sigma(X_k)<\infty$ for all $k\in\mathbb{N}$ and $X=\bigcup_{k\in\mathbb{N}}X_k$. We now set $B_n:=\bigcup_{k=0}^n X_k$. Then $B_0\subseteq B_1\subseteq B_2\subseteq\dots$ and $\bigcup_{n=0}^\infty B_n := \bigcup_{k=0}^\infty X_k = X$. Finally, since $\mu(X_k)<\infty$ for all $k\in\mathbb{N}$ we have $\mu(B_n)\leq\sum_{k=0}^n\mu(X_n)<\infty$.

(b) Show that there exist measurable sets $A_n \subseteq X$ with $\mu(A_n) < \infty$, $A_n \cap A_k = \emptyset$ if $n \neq k$ and $X = \bigcup_{n\in\mathbb{N}} A_n$.

Solution: Let B_n be sets such as those in (a) and define $A_0 := B_0$ and $A_{n+1} := B_{n+1} \setminus B_n$. Then $\mu(A_n) \leq \mu(B_n) < \infty$ for all $n \in \mathbb{N}$. Moreover $B_n = \bigcup_{k=1}^{\infty} A_n$, so $\bigcup_{k=0}^{\infty} A_k = \bigcup_{k=0}^{\infty} B_k = X$. Finally, if $j < k$, then $A_k \cap A_j = \emptyset$ since $A_j \subseteq B_j$ but $A_k \subseteq B_k \setminus B_j$.

6. Let μ be a measure defined on X. Show that μ is σ-finite if and only if there exists a measurable function $f\colon X \to (0,\infty)$ with $\int_X f\,d\mu < \infty$.

Solution: First suppose that μ is σ-finite. According to Question 5(b) there exist measurable sets $A_n \subseteq X$ with $\mu(A_n) < \infty$, $A_n \cap A_k = \emptyset$ if $n \neq k$ and $X = \bigcup_{n=1}^{\infty} A_n$. We let

$$f := \sum_{n=1} \frac{1}{2^n(1+\mu(A_n))} 1_{A_n}.$$

Then $f(x) > 0$ for all $x \in X$ and

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \frac{1}{2^n(1+\mu(A_n))}\mu(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

Hence f is a measurable function as required.

To prove the converse assume that $f\colon X \to (0,\infty)$ is such that $\int_X f\,d\mu < \infty$. For $n \in \mathbb{N}$ we set

$$X_n := \{x \in X\colon f(x) > 1/n\}.$$

Since $f(x) > 0$ for all $x \in X$ by assumption we have $X = \bigcup_{n\in\mathbb{N}} X_n$. Moreover

$$\mu(X_n) = n\int_{X_n} \frac{1}{n}\,d\mu \leq n\int_{X_n} f\,d\mu \leq n\int_X f\,d\mu < \infty$$

for all $n \in \mathbb{N}$. Hence μ is σ-finite.

7. A function $\varphi\colon (a,b) \to \mathbb{R}$ is called convex if $\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$ for all $x, y \in (a,b)$ and all $\lambda \in [0,1]$.

(a) Show that $\varphi\colon (a,b) \to \mathbb{R}$ is convex if and only if

$$\frac{\varphi(t)-\varphi(s)}{t-s} \leq \frac{\varphi(u)-\varphi(t)}{u-t} \tag{1}$$

whenever $a < s < t < u < b$.

Hint: Choose $\lambda \in (0,1)$ such that $t = (1-\lambda)s + \lambda u$.

Solution: Suppose that φ is convex. Choose $\lambda \in [0,1]$ such that $t = (1-\lambda)s + \lambda u$. Solving for λ we get

$$\lambda = \frac{t-s}{u-s}.$$

Hence by the convexity

$$\begin{aligned}\varphi(t) = \varphi((1-\lambda)s + \lambda u) &\leq (1-\lambda)\varphi(s) + \lambda\varphi(u)\\ &= \Big(1 - \frac{t-s}{u-s}\Big)\varphi(s) + \frac{t-s}{u-s}\varphi(u) = \varphi(s) + \frac{t-s}{u-s}\big(\varphi(u)-\varphi(s)\big).\end{aligned}$$

If we rearrange, then (1) follows.

Assume now that (1) holds. Let $s, u \in (a,b)$ such that $s < u$. Fix $\lambda \in (0,1)$ and set $t := (1-\lambda)s + \lambda u$. Doing the above calculation backwards we see that

$$\varphi((1-\lambda)s + \lambda u) = \varphi(t) \leq (1-\lambda)\varphi(s) + \lambda\varphi(u),$$

proving that φ is convex.

(b) Let $\varphi\colon (a,b) \to \mathbb{R}$ be convex. Use (1) to show that

$$\frac{\varphi(t)-\varphi(s)}{t-s} \le \frac{\varphi(u)-\varphi(v)}{u-v} \tag{2}$$

whenever $a < s < t < u < v < b$.

Solution: Using (1) twice we get

$$\frac{\varphi(t)-\varphi(s)}{t-s} \le \frac{\varphi(u)-\varphi(t)}{u-t} \le \frac{\varphi(v)-\varphi(u)}{v-u}.$$

(c) Show that a differentiable function $\varphi\colon (a,b) \to \mathbb{R}$ is convex if and only if φ' is increasing.

Solution: Let $s < t < u < v$. Then (2) holds. If we let $s \to t$ and $u \to v$ we get $\varphi'(t) \le \varphi'(v)$, so φ' is increasing.
Assume now that φ' is increasing. Let $s < t < u$. By the mean value theorem there exist $s \le \xi \le t$ and $t \le \eta \le u$ such that

$$\frac{\varphi(t)-\varphi(s)}{t-s} = \varphi'(\xi) \le \varphi'(\eta) = \frac{\varphi(u)-\varphi(t)}{u-t}$$

since $\xi \le \eta$ and φ' is increasing. By (a) it follows that φ is convex.

Challenge questions (optional)

8. Define

$$L^1_{\text{loc}}(\mathbb{R}) := \Big\{f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ measurable}, \int_B |f|\,dx < \infty \text{ for every } B \subseteq \mathbb{R} \text{ bounded}\Big\}.$$

Let $f \in L^1_{\text{loc}}(\mathbb{R})$. We say that f is *weakly differentiable* if there exists $g \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} g\varphi\,dx = -\int_{\mathbb{R}} f\varphi'\,dx$$

for all $\varphi \in C_c^\infty(\mathbb{R})$.

(a) If it exists, show that the weak derivative of a function is unique almost everywhere.

Solution: Suppose that g_1, g_2 are weak derivatives of f, then

$$\int_{\mathbb{R}} (g_1-g_2)\varphi\,dx = -\int_{\mathbb{R}} f\varphi'\,dx + \int_{\mathbb{R}} f\varphi'\,dx = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R})$. Hence we need to show that if $g \in L^1_{\text{loc}}(\mathbb{R})$ and

$$\int_{\mathbb{R}} g\varphi\,dx = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R})$, then $g = 0$ almost everywhere.

(b) If f is differentiable, show that $g = f'$ is the weak derivative of f.

Solution: Suppose that $\varphi \in C_c^\infty(\mathbb{R})$. Choose $a > 0$ such that $\operatorname{supp}\varphi \subseteq (-a,a)$. By integration by parts

$$\int_{\mathbb{R}} f'\varphi\,dx = \int_{-a}^{a} f'\varphi\,dx = f(x)\varphi(x)\Big|_{-a}^{a} - \int_{-a}^{a} f\varphi'\,dx = -\int_{-a}^{a} f\varphi'\,dx = \int_{\mathbb{R}} f\varphi'\,dx$$

since $\varphi(a) = \varphi(-a) = 0$ and $\operatorname{supp}\varphi' \subseteq \operatorname{supp}\varphi$. Hence f' must be the weak derivative by part (a).

(c) Compute the weak derivative of $f(x) = |x|$.

Solution: The given function is differentiable except at zero. Hence we will use integration by parts on the intervals $(-\infty, 0$ and $(0, \infty)$ to get the weak derivative. Hence let $\varphi \in C_c^\infty(\mathbb{R})$ and choose $a > 0$ such that $\operatorname{supp} \varphi \subseteq (-a, a)$. Then, using integration by parts

$$\begin{aligned}
\int_{\mathbb{R}} |x| \varphi'(x)\, dx &= -\int_{-a}^{0} x\varphi'(x)\, dx + \int_{0}^{a} x\varphi'(x)\, dx \\
&= -x\varphi(x)\Big|_{-a}^{0} + \int_{-a}^{0} \varphi(x)\, dx + x\varphi(x)\Big|_{0}^{a} - \int_{0}^{a} \varphi(x)\, dx \\
&= -\int_{-a}^{a} \operatorname{sign}(x)\varphi(x)\, dx = -\int_{\mathbb{R}} \operatorname{sign}(x)\varphi(x)\, dx
\end{aligned}$$

where $\operatorname{sign}(x) := x/|x|$ if $x \neq 0$ and $\operatorname{sign}(0) = 0$ is the sign of x. Hence the weak derivative of $|x|$ is $\operatorname{sign}(x)$.

(d) Show that $H(x) := 1_{[0,\infty)}$ is not weakly differentiable.

Solution: If $\varphi \in C_c^\infty(\mathbb{R})$, then

$$\int_{\mathbb{R}} H(x)\varphi'(x)\, dx = \int_{0}^{a} \varphi'(x)\, dx = \varphi(a) - \varphi(0) = -\varphi(0).$$

Except for one point, namely $x = 0$, the function H is differentiable, so an argument similar to that in part (a) shows that, if it exists, the weak derivative is zero. But that contradicts the above since in general

$$\varphi(0) \neq 0 = \int_{\mathbb{R}} 0\varphi\, dx.$$