The University of Sydney
School of Mathematics and Statistics

Tutorial 9 (Week 10)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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Material covered

(1) Definition and simple properties of the Fourier transform

(2) Applications of the Fourier transform

(3) Inversion formulae for the Fourier transform

Outcomes

After completing this tutorial you should

(1) should be familiar with the basic properties of the Fourier transform

(2) be able to prove simple properties of the Fourier transform

(3) have an idea on how to apply Fourier transforms to solve a simple partial differential equation

Questions to complete during the tutorial

1. Let $f \in \mathcal{L}^1(\mathbb{R})$. Express $\hat{g}(t)$ in terms of $\hat{f}(t)$, where

(a) $g(x) = f(x - x_0)$;

(b) $g(x) = f(cx)$, where $0 \neq c \in \mathbb{R}$;

(c) $g(x) = \overline{f(-x)}$.

2. Let $f \in \mathcal{L}^1(\mathbb{R}, \mathbb{R})$. Using the Riemann-Lebesgue Lemma, show that

$$\int_{\mathbb{R}} f(x)\cos(\lambda x)\,dx \to 0 \qquad \text{and} \qquad \int_{\mathbb{R}} f(x)\sin(\lambda x)\,dx \to 0$$

as $|\lambda| \to \infty$ ($\lambda \in \mathbb{R}$).

3. Let $f \in \mathcal{L}^1(\mathbb{R})$ such that $\int_{\mathbb{R}} |xf(x)|\,dx < \infty$. Show that the Fourier transform $\hat{f}$ of f is differentiable and that

$$\hat{f}'(t) = -2\pi i \int_{-\infty}^{\infty} xf(x)e^{-2\pi itx}\,dx$$

for all $t \in \mathbb{R}$.

Extra questions for further practice

4. Let $f \in \mathcal{L}^1(\mathbb{R})$. We have seen in Question 3 that if $\int_{-\infty}^{\infty} |xf(x)|\,dx < \infty$, then the Fourier transform $\hat{f}$ is differentiable.

(a) Show that if $\int_{-\infty}^{\infty} |x^2 f(x)|\,dx < \infty$, then the Fourier transform $\hat{f}$ is twice differentiable, and derive a formula for $\hat{f}''(t)$. Generalise to higher derivatives.

(b) What can you say about the differentiability of $\hat{f}$ if $\int_{-\infty}^{\infty} |x^n f(x)|\,dx < \infty$ for some $n \in \mathbb{N}$?

5. (a) Suppose that $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}) \cap C^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$. Show that $\widehat{f'} = 2\pi i t \hat{f}$. This means differentiation is turned into multiplication by the Fourier transform.

(b) Generalise the above to higher dimensions. We assume that $f \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ with partial derivatives in $L^1(\mathbb{R}^N)$. Show that

$$\widehat{\frac{\partial f}{\partial x_k}} = 2\pi i t_k \hat{f}.$$

6. Consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 \qquad & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x,0) = u_0(x) \quad & \text{in } \mathbb{R}^N \end{aligned}$$

This question guides you through a solution to the heat equation using Fourier transforms.

(a) Assuming that u is a sufficiently nice solution, show that

$$\frac{\partial}{\partial t}\hat{u}(\xi, t) + 4\pi^2|\xi|^2\hat{u}(\xi,t) = 0, \qquad \hat{u}(\xi, 0) = \hat{u}_0(\xi)$$

is the Fourier transform of the equation with respect to the x-variable.

(b) Solve the ordinary differential equation in the previous part to find $\hat{u}(\xi, t)$ for $(\xi, t) \in \mathbb{R}^N \times (0, \infty)$.

(c) Show that the solution to the heat equation has the form $u(x,t) = (g_t * u_0)(x)$, where $g_t(x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}$. You may assume that $\hat{u}_0 \in L^1(\mathbb{R}^N)$. ($g_t$ is called the *heat kernel*.)

7. Let $g \colon \mathbb{R} \to \mathbb{R}$ be the odd function which satisfies $g(x) = e^{-x}$ if $x > 0$ (and $g(0) = 0$). Calculate the Fourier transform $\hat{g}(t)$ of g, and verify that it is odd.

***8.** Suppose that $f \in \mathcal{L}^1(\mathbb{R})$ is differentiable at a point x_0. Show that

$$f(x_0) = \lim_{A \to \infty} \int_{-A}^{A} \hat{f}(t) e^{2\pi i t x_0}\, dt.$$

Hint: First look at the case $x_0 = 0$ and $f(0) = 0$. For the general case consider $h(x) = f(x + x_0) - f(x_0)e^{-\pi x^2}$.

Challenge questions (optional)

9. Generalise the result in Question 8 as follows: instead of assuming that f is differentiable at x_0, assume merely that $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ and $f(x_0^-) = \lim_{x \to x_0^-} f(x)$ exist and that the left and right slopes

$$m_R = \lim_{x \to x_0^+} \frac{f(x) - f(x_0^+)}{x - x_0} \quad \text{and} \quad m_L = \lim_{x \to x_0^-} \frac{f(x) - f(x_0^-)}{x - x_0}$$

exist. Show now that

$$\int_{-A}^{A} \hat{f}(t) e^{2\pi i t x_0}\, dt \to \frac{f(x_0^+) + f(x_0^-)}{2}$$

as $A \to \infty$.

Hint: Define a function h by setting $h(0) = 0$ and $h(x) = f(x_0^+ x) - \ell e^{-\pi x^2} - c g(x)$ for $x \neq 0$, where $g(x)$ is as in Question 7, $\ell = (f(x_0^+) + f(x_0^-))/2$ and $c = (f(x_0^+) - f(x_0^-))/2$. Apply Question 8 to the function $h(x)$.