THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 9 (Week 10)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
Lecturer: Daniel Daners

Material covered

(1) Definition and simple properties of the Fourier transform

(2) Applications of the Fourier transform

(3) Inversion formulae for the Fourier transform

Outcomes

After completing this tutorial you should

(1) should be familiar with the basic properties of the Fourier transform

(2) be able to prove simple properties of the Fourier transform

(3) have an idea on how to apply Fourier transforms to solve a simple partial differential equation

Questions to complete during the tutorial

1. Let $f \in \mathcal{L}^1(\mathbb{R})$. Express $\hat{g}(t)$ in terms of $\hat{f}(t)$, where

(a) $g(x) = f(x - x_0)$;

Solution: By definition of the Fourier transform and the substitution $u = x - x_0$

$$\hat{g}(t) = \int_{\mathbb{R}^N} f(x-x_0)e^{-2\pi i x\cdot t}\,dx = e^{-2\pi i x_0\cdot t}\int_{\mathbb{R}^N} f(x-x_0)e^{-2\pi i(x-x_0)\cdot t}\,dx$$
$$= e^{-2\pi i x_0\cdot t}\int_{\mathbb{R}^N} f(u)e^{-2\pi i u\cdot t}\,du = e^{-2\pi i x_0\cdot t}\hat{f}(t).$$

(b) $g(x) = f(cx)$, where $0 \neq c \in \mathbb{R}$;

Solution: By definition of the Fourier transform and the substitution $u = cx$

$$\hat{g}(t) = \int_{\mathbb{R}^N} f(cx)e^{-2\pi i x\cdot t}\,dx = \int_{\mathbb{R}^N} f(cx)e^{-2\pi i (cx)\cdot(t/c)}\,dx$$
$$= \frac{1}{|c|^N}\int_{\mathbb{R}^N} f(u)e^{-2\pi i u\cdot(t/c)}\,du = \frac{1}{|c|^N}\hat{f}\big(\frac{t}{c}\big)$$

(c) $g(x) = \overline{f(-x)}$.

Solution: By definition of the Fourier transform and the integral of a complex valued function

$$\hat{g}(t) = \int_{\mathbb{R}^N} \overline{f(-x)}e^{-2\pi i x\cdot t}\,dx = \int_{\mathbb{R}^N} \overline{f(x)}e^{2\pi i x\cdot t}\,dx$$
$$= \int_{\mathbb{R}^N} \overline{f(x)e^{-2\pi i x\cdot t}}\,dx = \overline{\int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot t}\,dx} = \overline{\hat{f}(t)}.$$

2. Let $f \in \mathcal{L}^1(\mathbb{R}, \mathbb{R})$. Using the Riemann-Lebesgue Lemma, show that

$$\int_{\mathbb{R}} f(x)\cos(\lambda x)\,dx \to 0 \qquad \text{and} \qquad \int_{\mathbb{R}} f(x)\sin(\lambda x)\,dx \to 0$$

as $|\lambda| \to \infty$ ($\lambda \in \mathbb{R}$).

Solution: Note that

$$\hat{f}(\lambda/2\pi) = \int_{\mathbb{R}} f(x)e^{ix\lambda}\,dx = \int_{\mathbb{R}} f(x)\cos(\lambda x)\,dx + i\int_{\mathbb{R}} f(x)\sin(\lambda x)\,dx.$$

By the Riemann-Lebesgue Lemma $\hat{f}(\lambda/2\pi) \to 0$ as $|\lambda| \to \infty$ and therefore the real and imaginary parts converge to zero, that is,

$$\int_{\mathbb{R}} f(x)\cos(\lambda x)\,dx \to 0 \qquad \text{and} \qquad \int_{\mathbb{R}} f(x)\sin(\lambda x)\,dx \to 0$$

as $|\lambda| \to \infty$.

3. Let $f \in \mathcal{L}^1(\mathbb{R})$ such that $\int_{\mathbb{R}} |xf(x)|\,dx < \infty$. Show that the Fourier transform $\hat{f}$ of f is differentiable and that

$$\hat{f}'(t) = -2\pi i \int_{-\infty}^{\infty} xf(x)e^{-2\pi itx}\,dx$$

for all $t \in \mathbb{R}$.

Solution: We use the theorem on the differentiation of parameter integrals setting $f(x,t) = f(x)e^{-2\pi itx}$ and $g(x) = 2\pi|xf(x)|$. Now

$$\frac{\partial f}{\partial t}(x,t) = -2\pi ixf(x)e^{-2\pi itx},$$

and so the inequality

$$\Big|\frac{\partial f}{\partial t}(x,t)\Big| \le |g(x)|$$

and the hypothesis that $|xf(x)|$ be integrable show that the theorem on the differentiation of parameter integrals is applicable.

Extra questions for further practice

4. Let $f \in \mathcal{L}^1(\mathbb{R})$. We have seen in Question 3 that if $\int_{-\infty}^{\infty} |xf(x)|\,dx < \infty$, then the Fourier transform $\hat{f}$ is differentiable.

(a) Show that if $\int_{-\infty}^{\infty} |x^2 f(x)|\,dx < \infty$, then the Fourier transform $\hat{f}$ is twice differentiable, and derive a formula for $\hat{f}''(t)$. Generalise to higher derivatives.

Solution: We apply the theorem on the differentiation of parameter integrals. We set $f(x,t) = -2\pi ixf(x)e^{-2\pi itx}$ and $g(x) = 4\pi^2x^2|f(x)|$. Because $|x| \le \frac{1}{2}(1+x^2)$ for all x, the hypotheses $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ and $\int_{-\infty}^{\infty} x^2|f(x)|\,dx < \infty$ imply that $\int_{-\infty}^{\infty} |xf(x)|\,x < \infty$. Hence $f(x,t)$ is an integrable function of x for each fixed t, and by Question 3 $\hat{f}$ is differentiable. Now

$$\frac{\partial f}{\partial t}(x,t) = -4\pi^2x^2f(x)e^{-2\pi itx},$$

and so the inequality

$$\Big|\frac{\partial f}{\partial t}(x,t)\Big| \le |g(x)|$$

holds. By applying the theorem on the differentiation of parameter integrals we get

$$(\hat{f}\,)''(t) = -4\pi^2 \int_{-\infty}^{\infty} x^2f(x)e^{-2\pi itx}\,dx.$$

(b) What can you say about the differentiability of $\hat{f}$ if $\int_{-\infty}^{\infty} |x^n f(x)|\, dx < \infty$ for some $n \in \mathbb{N}$?

Solution: If $\int_{-\infty}^{\infty} |x^n f(x)|\, dx < \infty$, then $\int_{-\infty}^{\infty} |x^k f(x)|\, dx < \infty$ for $k = 0, 1, \ldots, n$ (because $|x^k| \leq 1$ if $|x| \leq 1$, and $|x^k| \leq |x^n|$ if $|x| \geq 1$, so that $|x^k f(x)| \leq (1 + |x^n|)|f(x)|$ for all x). A routine induction shows that $\hat{f}$ is n times differentiable on $\mathbb{R}$, and that

$$\hat{f}^{(n)}(t) = (-2\pi i)^n \int_{-\infty}^{\infty} x^n f(x) e^{-2\pi i t x}\, dx$$

for all $t \in \mathbb{R}$. Note that $\hat{f}^{(n)}$ is the Fourier transform of the integrable function $x \mapsto (-2\pi i)^n x^n f(x)$, and so is continuous by the Riemann-Lebesgue lemma. The above formula shows that $\hat{f}$ is very smooth if f decays very fast.

5. (a) Suppose that $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}) \cap C^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$. Show that $\widehat{f'} = 2\pi i t \hat{f}$. This means differentiation is turned into multiplication by the Fourier transform.

Solution: Using integration by parts we have

$$\widehat{f'}(t) = \int_{\mathbb{R}} f'(x) e^{-2\pi i x t}\, dx = 2\pi i t \int_{\mathbb{R}} f(x) e^{2\pi i x t}\, dx = 2\pi i t \hat{f}(t)$$

(b) Generalise the above to higher dimensions. We assume that $f \in L^1(\mathbb{R}^N) \cap C_0(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ with partial derivatives in $L^1(\mathbb{R}^N)$. Show that

$$\widehat{\frac{\partial f}{\partial x_k}} = 2\pi i t_k \hat{f}.$$

Solution: This is analogous to the previous part using Fubini's theorem and then do an integration by parts in the x_k-coordinate.

6. Consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = 0 \qquad & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) \quad & \text{in } \mathbb{R}^N \end{aligned}$$

This question guides you through a solution to the heat equation using Fourier transforms.

(a) Assuming that u is a sufficiently nice solution, show that

$$\frac{\partial}{\partial t} \hat{u}(\xi, t) + 4\pi^2 |\xi|^2 \hat{u}(\xi, t) = 0, \qquad \hat{u}(\xi, 0) = \hat{u}_0(\xi)$$

is the Fourier transform of the equation with respect to the x-variable.

Solution: This follows from Question 5 if we note that

$$\widehat{\frac{\partial^2 u}{\partial x_k^2}} = (2\pi i)^2 \xi_k^2 \hat{u} = -4\pi \xi_k^2 \hat{u}.$$

Adding up we get $\widehat{\Delta u} = -4\pi |\xi|^2 \hat{u}$. (Since t is the time variable in this example we use ξ rather than t for the Fourier transform.)

(b) Solve the ordinary differential equation in the previous part to find $\hat{u}(\xi, t)$ for $(\xi, t) \in \mathbb{R}^N \times (0, \infty)$.

Solution: The solution to the (separable) differential equation is

$$\hat{u}(\xi, t) = e^{-4\pi^2 |\xi|^2 t} \hat{u}_0(\xi).$$

(ξ is just a parameter for this differential equation.)

(c) Show that the solution to the heat equation has the form $u(x,t) = (g_t * u_0)(x)$, where $g_t(x) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}$. You may assume that $\hat{u}_0 \in L^1(\mathbb{R}^N)$. ($g_t$ is called the *heat kernel.*)

Solution: The solution $\hat{u}(\xi,t)$ is a product of Fourier transforms, and therefore the Fourier transform of a convolution $\widehat{g_t * u_0}(\xi)$. We have to find g_t so that

$$\hat{g}_t(\xi) = e^{-4\pi^2 t|\xi|^2}.$$

Since the latter function is in $L^1(\mathbb{R}^N)$ the inverse Fourier transform is given by

$$g_t(x) = \int_{\mathbb{R}^N} e^{-4\pi^2 t|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi.$$

If we do the substitution $y = \sqrt{4\pi t}\xi$, then

$$g_t(x) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} e^{-\pi|y|^2} e^{2\pi i y\cdot\frac{x}{\sqrt{4\pi t}}}\, dy.$$

Setting $\varphi = e^{-\pi|x|^2}$ we know that $\hat{\varphi} = \varphi$. Hence the inverse Fourier transform of φ is φ as well. Hence

$$g_t(x) = \varphi\Big(\frac{x}{\sqrt{4\pi t}}\Big) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}.$$

Remark: The heat kernel is like the normal distribution. This is no accident as heat at a molecular level is caused by diffusion. Diffusion is a random process, called Brownian motion. Molecules move along a random path and the probability to move from x to y in time t is normally distributed by g_t. In the theory of stochastic processes, the heat kernel is therefore called the *transition probability.* Hence there are interesting connections between partial differential equations and probability theory.

7. Let $g\colon \mathbb{R} \to \mathbb{R}$ be the odd function which satisfies $g(x) = e^{-x}$ if $x > 0$ (and $g(0) = 0$). Calculate the Fourier transform $\hat{g}(t)$ of g, and verify that it is odd.

Solution: Since $g(x)$ is odd,

$$\begin{aligned}\hat{g}(t) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i t x}\, dx = -i\int_{-\infty}^{\infty} g(x)\sin(2\pi t x)\, dx \\ = -2i\int_0^{\infty} g(x)\sin(2\pi t x)\, dx = -2i\int_0^{\infty} e^{-x}\sin(2\pi t x)\, dx.\end{aligned}$$

Hence

$$\begin{aligned}\hat{g}(t) &= -2i\,\mathrm{Im}\int_0^{\infty} e^{-x}e^{2\pi i t x}\, dx = -2i\,\mathrm{Im}\int_0^{\infty} e^{(-1+2\pi i t)x}\, dx \\ &= -2i\,\mathrm{Im}\,\frac{e^{(-1+2\pi i t)x}}{(-1+2\pi i t)}\Big|_{x=0}^{x=\infty} = -2i\,\mathrm{Im}\,\frac{1}{(1-2\pi i t)} = -2i\,\mathrm{Im}\,\frac{1+2\pi i t}{(1+4\pi^2 t^2)} = \frac{-4\pi i t}{1+4\pi^2 t^2}.\end{aligned}$$

In particular $\hat{g}(t)$ is odd.

***8.** Suppose that $f \in \mathcal{L}^1(\mathbb{R})$ is differentiable at a point x_0. Show that

$$f(x_0) = \lim_{A\to\infty}\int_{-A}^{A} \hat{f}(t)e^{2\pi i t x_0}\, dt.$$

Hint: First look at the case $x_0 = 0$ and $f(0) = 0$. For the general case consider $h(x) = f(x+x_0) - f(x_0)e^{-\pi x^2}$.

Solution: First suppose that $x_0 = 0$ and that $f(0) = 0$. Then we have to show that

$$\int_{-A}^{A} \hat{f}(t)\,dt \to 0 \quad \text{as } A \to \infty. \tag{1}$$

By using that $\widehat{1_{[-A,A]}} = \frac{\sin(2\pi A x)}{\pi x}$ we get

$$\int_{-A}^{A} \hat{f}(t)\,dt = \int_{-\infty}^{\infty} \hat{f}(t) 1_{[-A,A]}(t)\,dt = \int_{-\infty}^{\infty} f(x)\widehat{1_{[-A,A]}}(x)\,dx = \int_{-\infty}^{\infty} f(x)\frac{\sin(2\pi A x)}{\pi x}\,dx.$$

Since $g(x) = f(x)/x \to f'(0)$ as $x \to 0$ and $|f(x)/x| \leq f(x)$ for $|x| \geq 1$ the function $g(x)$ is integrable. Hence

$$\int_{-A}^{A} \hat{f}(t)\,dt = \int_{-\infty}^{\infty} f(x)\frac{\sin(2\pi A x)}{\pi x}\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\sin(2\pi A x)\,dx \to 0$$

as $A \to \infty$ by Question 2.

Now we drop the extra hypotheses that $x_0 = 0$ and that $f(0) = 0$. Consider

$$h(x) = f(x + x_0) - f(x_0)e^{-\pi x^2}.$$

It is clear that h is in $\mathcal{L}^1(\mathbb{R})$, that $h(0) = 0$ and that $h'(0)$ exists. Hence by (1), with h in place of f, we have

$$\int_{-A}^{A} \hat{h}(t)\,dt \to 0$$

as $A \to \infty$. But by Exercise 1 and using that $e^{-\pi x^2}$ is its own Fourier transform

$$\hat{h}(t) = \hat{f}(t)e^{2\pi i t x_0} - f(x_0)e^{-\pi t^2},$$

and so

$$\int_{-A}^{A} \hat{f}(t)e^{2\pi i t x_0}\,dt = \int_{-A}^{A} \hat{h}(t)\,dt + f(x_0)\int_{-A}^{A} e^{-\pi t^2}\,dt \to 0 + f(x_0)$$

as $A \to \infty$.

Challenge questions (optional)

9. Generalise the result in Question 8 as follows: instead of assuming that f is differentiable at x_0, assume merely that $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ and $f(x_0^-) = \lim_{x \to x_0^-} f(x)$ exist and that the left and right slopes

$$m_R = \lim_{x \to x_0^+} \frac{f(x) - f(x_0^+)}{x - x_0} \quad \text{and} \quad m_L = \lim_{x \to x_0^-} \frac{f(x) - f(x_0^-)}{x - x_0}$$

exist. Show now that

$$\int_{-A}^{A} \hat{f}(t)e^{2\pi i t x_0}\,dt \to \frac{f(x_0^+) + f(x_0^-)}{2}$$

as $A \to \infty$.

Hint: Define a function h by setting $h(0) = 0$ and $h(x) = f(x_0^+ x) - \ell e^{-\pi x^2} - cg(x)$ for $x \neq 0$, where $g(x)$ is as in Question 7, $\ell = (f(x_0^+) + f(x_0^-))/2$ and $c = (f(x_0^+) - f(x_0^-))/2$. Apply Question 8 to the function $h(x)$.

Solution: Note that in the first part of the proof in Question 8 we only really need that $f(0) = 0$, that f is continuous at zero and that $f(x)/x$ has left and right limits, so that $f(x)/x$ is integrable.

Now let $g(x)$ be the function from Question 7. Then $g(x) \to 1$ as $x \to 0$ from the right, and $g(x) \to -1$ as $x \to 0$ from the left. So the function $h(x)$ of the hint satisfies

$$\lim_{x \to 0^+} h(x) = f(x_0^+) - \ell - c = 0$$

and

$$\lim_{x \to 0^-} h(x) = f(x_0^-) - \ell + c = 0.$$

Since we have defined $h(0) = 0$, it follows that h is continuous at 0. Also, if $x > 0$, using $h(0) = 0 = f(x_0^+) - \ell - c$, we get

$$\frac{h(x) - h(0)}{x} = \frac{f(x) - f(x_0^+)}{x} - \ell \frac{e^{-\pi x^2} - 1}{x} - c\, e^{-x} - 1x,$$

and this tends to $m_R + c$ as $x \to 0^+$. Similarly, if $x < 0$, using $h(0) = 0 = f(x_0^-) - \ell + c$, we get

$$\frac{h(x) - h(0)}{x} = \frac{f(x) - f(x_0^-)}{x} - \ell \frac{e^{-\pi x^2} - 1}{x} - c\, e^{x} + 1x,$$

and this tends to $m_L - c$ as $x \to 0^-$.

Hence we have shown that h is continuous at 0, has left and right derivatives at zero and so by the introductory remarks

$$\int_{-A}^{A} \hat{h}(t)\, dt \to 0$$

as $A \to \infty$. On the other hand, by definition of h

$$\hat{h}(t) = \hat{f}(t) e^{2\pi i t x_0} - \ell e^{-\pi t^2} - c \hat{g}(t)$$

and so if we use that $\hat{g}$ is odd

$$\begin{aligned} \int_{-A}^{A} \hat{f}(t) e^{2\pi i t x_0}\, dt &= \int_{-A}^{A} \hat{h}(t)\, dt + \ell \int_{-A}^{A} e^{-\pi t^2}\, dt + c \int_{-A}^{A} \hat{g}(t)\, dt \\ &= \int_{-A}^{A} \hat{h}(t)\, dt + \ell \int_{-A}^{A} e^{-\pi t^2}\, dt + 0 \to 0 \end{aligned}$$

as $A \to \infty$ and the result is proved.