The University of Sydney
School of Mathematics and Statistics

Tutorial 10 (Week 11)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
Lecturer: Daniel Daners

Material covered

(1) The Fourier transform on $L^2(\mathbb{R}^N)$

(2) Applications of Plancherel's theorem.

(3) The Riemann-Lebesgue Lemma.

(4) The Fourier inversion theorem on $L^2(\mathbb{R}^N)$.

Outcomes

After completing this tutorial you should

(1) be able to work with the Fourier transform and Plancherel's theorem on $L^2(\mathbb{R}^N)$.

(2) work with the Fourier inversion formula in various contexts.

Questions to complete during the tutorial

1. We know from lectures that $\widehat{1_{[-1,1]}}(t) = \dfrac{\sin(2\pi t)}{\pi t}$. Use Plancherel's theorem to prove that

$$\int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^2 dx = \pi.$$

2. Let $f \in L^1(\mathbb{R}^N, \mathbb{C})$. In lectures we proved the inversion formula

$$\int_{\mathbb{R}^N} \hat{f}(t) e^{2\pi i x\cdot t} e^{-\pi|t|^2/n^2}\, dt \to f$$

in $L^1(\mathbb{R}^N, \mathbb{C})$. Suppose now that $f, \hat{f} \in L^1(\mathbb{R}^N, \mathbb{C})$.

(a) Use the above inversion formula to show that

$$f(x) = \int_{\mathbb{R}^N} \hat{f}(t) e^{2\pi i x\cdot t}\, dt$$

almost everywhere.

(b) Set $g(x) := \hat{f}(-x)$. Show that $f = \hat{g}$ almost everywhere and therefore $f \in C_0(\mathbb{R}^N)$ when possibly modified on a set of measure zero.

3. (a) Compute the Fourier transform of $f(x) := e^{-|x|}$ as a function on $\mathbb{R}$.

(b) Use the Fourier transform of $e^{-|x|}$ and the inversion formula from Question **2** to compute the Fourier transform of $g(x) = \dfrac{1}{1+x^2}$.

Extra questions for further practice

4. Use the inversion theorem to characterise functions $f \in L^1(\mathbb{R}^N, \mathbb{C})$ with the following properties:

(a) $\hat{f}$ is even. (b) $\hat{f}$ is odd. (c) $\hat{f}$ is real valued.

5. Tutorial 9, Question **5** tells us that $\widehat{\partial f/\partial x_k}(t) = 2\pi i t_k \hat{f}(t)$. Given a linear differential operator L such as the Laplace operator we can compute the Fourier transform $\widehat{Lu}$ which the multiplication of $\hat{u}$ with a polynomial. That polynomial is called the *symbol* of L. For $s \geq 0$ define

$$H^s(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N) \colon (1 + |t|^2)^{s/2}\hat{u} \in L_2(\mathbb{R}^N)\big\}.$$

This is called the *Sobolev space* of order s. Sobolev spaces are widely used in the modern theory and application of partial differential equations.

(a) Show that $H^{s_2}(\mathbb{R}^N) \subseteq H^{s_1} \subseteq L_2(\mathbb{R}^N)$ whenever $0 \leq s_1 \leq s_2$.

(b) Show that $H^s(\mathbb{R}^N) \subseteq C_0(\mathbb{R}^N)$ if $s > N/2$ (*Sobolev embedding theorem*).
Hint: Show that $\hat{u} \in L^1(\mathbb{R}^N)$ and then use Question 2.

(c) Show that $H^s(\mathbb{R}^N) \subseteq C_0^k(\mathbb{R}^N)$ if $s > k + N/2$ (*general Sobolev embedding theorem*). (This is based on the principle that if the Fourier transform decays fast, then the original function was quite regular and vice versa.)

(d) Compute the symbol of the Laplace operator and also $u - \Delta u$.

(e) Prove that $u \to \lambda u - \Delta u$ is a bijection from $H^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ for all $\lambda > 0$.

(f) Prove that $\to \lambda u - \Delta u$ is a bijection from $H^{s+2}(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$ for all $\lambda > 0$, that is, the partial differential equation $\lambda u - \Delta u = f$ has a unique solution $u \in H^{2+s}(\mathbb{R}^N)$ for all $f \in H^s(\mathbb{R}^N)$.

(g) Let $f \in C_c^\infty(\mathbb{R}^N)$. Show that $\lambda u - \Delta u = f$ has a unique solution in $C_0^\infty(\mathbb{R}^N)$ for all $\lambda > 0$.

Challenge questions (optional)

6. Consider the space

$$\mathcal{S}(\mathbb{R}) := \{x \in C^\infty(\mathbb{R}) \colon |x|^k |f^{(n)}(x)| \to 0 \text{ as } |x| \to \infty \text{ for all } k, n \in \mathbb{N}\}.$$

This is called the *Schwartz class* of smooth very fast decaying functions.

(a) Show that $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(b) If $f \in \mathcal{S}(\mathbb{R})$, show that $\hat{f} \in \mathcal{S}(\mathbb{R})$.

(c) Show that the Fourier transform $\mathcal{F} \colon \mathcal{S} \to \mathcal{S}$ is a bijection.