The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 10 (Week 11)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
Lecturer: Daniel Daners

Material covered

(1) The Fourier transform on $L^2(\mathbb{R}^N)$

(2) Applications of Plancherel's theorem.

(3) The Riemann-Lebesgue Lemma.

(4) The Fourier inversion theorem on $L^2(\mathbb{R}^N)$.

Outcomes

After completing this tutorial you should

(1) be able to work with the Fourier transform and Plancherel's theorem on $L^2(\mathbb{R}^N)$.

(2) work with the Fourier inversion formula in various contexts.

Questions to complete during the tutorial

1. We know from lectures that $\widehat{1_{[-1,1]}}(t) = \dfrac{\sin(2\pi t)}{\pi t}$. Use Plancherel's theorem to prove that

$$\int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^2 dx = \pi.$$

Solution: By Plancherel's theorem

$$2 = \int_{\mathbb{R}} 1_{[-1,1]}^2 \, dt = \int_{\mathbb{R}} \left|\frac{\sin 2\pi t}{\pi t}\right|^2 dt.$$

If we make the substitution $x = 2\pi t$ we get

$$2 = \int_{\mathbb{R}} \left|\frac{\sin x}{\pi \frac{x}{2\pi}}\right|^2 \frac{1}{2\pi} \, dt = \frac{2}{\pi} \int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^2 dx.$$

Multiplying by $\pi/2$ we get the required integral.

2. Let $f \in L^1(\mathbb{R}^N, \mathbb{C})$. In lectures we proved the inversion formula

$$\int_{\mathbb{R}^N} \hat{f}(t) e^{2\pi i x \cdot t} e^{-\pi |t|^2/n^2} \, dt \to f$$

in $L^1(\mathbb{R}^N, \mathbb{C})$. Suppose now that $f, \hat{f} \in L^1(\mathbb{R}^N, \mathbb{C})$.

(a) Use the above inversion formula to show that

$$f(x) = \int_{\mathbb{R}^N} \hat{f}(t) e^{2\pi i x \cdot t} \, dt$$

almost everywhere.

Solution: Note that

$$|\hat{f}(t)e^{2\pi i x\cdot t}e^{-\pi|t|^2/n^2}| \leq |\hat{f}(t)|$$

for all $t \in \mathbb{R}^N$. Since $\hat{f} \in L^1(\mathbb{R}^N, \mathbb{C})$ the dominated convergence theorem implies that

$$\int_{\mathbb{R}^N} \hat{f}(t)e^{2\pi i x\cdot t}e^{-\pi|t|^2/n^2}\,dt \to \int_{\mathbb{R}^N} \hat{f}(t)e^{2\pi i x\cdot t}\,dt$$

for all $t \in \mathbb{R}^N$. Together with the inversion formlua it follows that

$$\int_{\mathbb{R}^N} \hat{f}(t)e^{2\pi i x\cdot t}e^{-\pi|t|^2/n^2}\,dt \to \int_{\mathbb{R}^N} \hat{f}(t)e^{2\pi i x\cdot t}\,dt = f$$

in $L^1(\mathbb{R}^N)$. In particular the two limits are equal almost everywhere.

(b) Set $g(x) := \hat{f}(-x)$. Show that $f = \hat{g}$ almost everywhere and therefore $f \in C_0(\mathbb{R}^N)$ when possibly modified on a set of measure zero.

Solution: By the formula in (a) we have, using the substitution $y = -x$,

$$\hat{g}(t) = \int_{\mathbb{R}^N} g(x)e^{-2\pi i x\cdot t}\,dx = \int_{\mathbb{R}^N} \hat{f}(-x)e^{-2\pi i x\cdot t}\,dx = \int_{\mathbb{R}^N} \hat{f}(y)e^{2\pi i y\cdot t}\,dy = f(t)$$

almost everywhere. Thre remaining assertion follows from the Riemann-Lebesgue Lemma.

3. (a) Compute the Fourier transform of $f(x) := e^{-|x|}$ as a function on $\mathbb{R}$.

Solution: By definition of the Fourier transform and since $e^{-(1\pm 2\pi i t)x} \to 0$ as $x \to \infty$ we have

$$\begin{aligned}
\hat{f}(t) &= \int_{\mathbb{R}} e^{-|x|}e^{-2\pi i x t}\,dx = \int_0^\infty e^{-x}e^{-2\pi i x t}\,dx + \int_{-\infty}^0 e^{x}e^{-2\pi i x t}\,dx \\
&= \int_0^\infty e^{-x}e^{-2\pi i x t}\,dx + \int_0^\infty e^{-x}e^{2\pi i x t}\,dx = \int_0^\infty e^{-(1+2\pi i x t)x} + e^{-(1-2\pi i x t)}\,dx \\
&= -\frac{1}{1+2\pi i x t}e^{-(1+2\pi i x t)x}\Big|_0^\infty - \frac{1}{1-2\pi i x t}e^{-(1-2\pi i x t)}\Big|_0^\infty \\
&= \frac{1}{1+2\pi i x t} + \frac{1}{1-2\pi i x t} = \frac{2}{1+4\pi^2 t^2}.
\end{aligned}$$

(b) Use the Fourier transform of $e^{-|x|}$ and the inversion formula from Question 2 to compute the Fourier transform of $g(x) = \dfrac{1}{1+x^2}$.

Solution: Clearly $g(x) := \dfrac{1}{1+x^2}$ and therefore $\dfrac{2}{1+4\pi^2 t^2}$ are integrable. Hence from the inversion formula in Question 2 and the substitution $y = 2\pi t$

$$e^{-|x|} = \int_{\mathbb{R}} \frac{2}{1+4\pi^2 t^2}e^{2\pi i x t}\,dt = \int_{\mathbb{R}} \frac{2}{1+y^2}e^{-2\pi i y(-x/2\pi)}\frac{1}{2\pi}\,dy = \frac{1}{\pi}\hat{g}(-x/2\pi).$$

Setting $t := -x/2\pi$ we get $e^{-2\pi|t|} = \hat{g}(t)/\pi$ and so

$$\hat{g}(t) = \pi e^{-2\pi|t|}$$

for all $t \in \mathbb{R}$. The formula is valid for all t because all functions involved are continuous.

Extra questions for further practice

4. Use the inversion theorem to characterise functions $f \in L^1(\mathbb{R}^N, \mathbb{C})$ with the following properties:

(a) $\hat{f}$ is even.

Solution: Assume that $\hat{f}$ is even, that is,

$$\int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot t}\,dt = \hat{f}(t) = \hat{f}(-t) = \int_{\mathbb{R}^N} f(x)e^{2\pi i x\cdot t}\,dt = \int_{\mathbb{R}^N} f(-x)e^{-2\pi i x\cdot t}\,dt$$

for all $t \in \mathbb{R}^N$. Hence

$$0 = \int_{\mathbb{R}^N} \big(f(x) - f(-x)\big)e^{-2\pi i x\cdot t}\,dt$$

for all $t \in \mathbb{R}^N$. Hence by the injectivity of the Fourier transform $\hat{f}$ is even if and only if $f(x) = f(-x)$ almost everywhere, that is, f is even almost everywhere.

(b) $\hat{f}$ is odd.

Solution: Assume that $\hat{f}$ is odd, that is,

$$\int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot t}\,dt = \hat{f}(t) = -\hat{f}(-t) = -\int_{\mathbb{R}^N} f(x)e^{2\pi i x\cdot t}\,dt = -\int_{\mathbb{R}^N} f(-x)e^{-2\pi i x\cdot t}\,dt$$

for all $t \in \mathbb{R}^N$. Hence

$$0 = \int_{\mathbb{R}^N} \big(f(x) + f(-x)\big)e^{-2\pi i x\cdot t}\,dt$$

for all $t \in \mathbb{R}^N$. Hence by the injectivity of the Fourier transform $\hat{f}$ is odd if and only if $f(x) = -f(-x)$ almost everywhere, that is, f is odd almost everywhere.

(c) $\hat{f}$ is real valued.

Solution: Assume that $\hat{f}$ is real valued, that is,

$$\begin{aligned}\int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot t}\,dt = \hat{f}(t) = \overline{\hat{f}(t)} &= \overline{\int_{\mathbb{R}^N} f(x)e^{2\pi i x\cdot t}\,dt} \\ &= \int_{\mathbb{R}^N} \overline{f(x)}e^{2\pi x\cdot t}\,dt = \int_{\mathbb{R}^N} \overline{f(-x)}e^{-2\pi i x\cdot t}\,dt\end{aligned}$$

for all $t \in \mathbb{R}^N$. Hence

$$0 = \int_{\mathbb{R}^N} \big(f(x) - \overline{f(-x)}\big)e^{-2\pi i x\cdot t}\,dt$$

for all $t \in \mathbb{R}^N$. Hence by the injectivity of the Fourier transform $\hat{f}$ is real valued if and only if $f(x) = \overline{f(-x)}$ almost everywhere.

5. Tutorial 9, Question **5** tells us that $\widehat{\partial f/\partial x_k}(t) = 2\pi i t_k \hat{f}(t)$. Given a linear differential operator L such as the Laplace operator we can compute the Fourier transform $\widehat{Lu}$ which the multiplication of $\hat{u}$ with a polynomial. That polynomial is called the *symbol* of L. For $s \geq 0$ define

$$H^s(\mathbb{R}^N) := \big\{u \in L^2(\mathbb{R}^N)\colon (1 + |t|^2)^{s/2}\hat{u} \in L_2(\mathbb{R}^N)\big\}.$$

This is called the *Sobolev space* of order s. Sobolev spaces are widely used in the modern theory and application of partial differential equations.

(a) Show that $H^{s_2}(\mathbb{R}^N) \subseteq H^{s_1} \subseteq L_2(\mathbb{R}^N)$ whenever $0 \leq s_1 \leq s_2$.

Solution: As $(1 + |t|^2)^{s/2}$ is increasing in $s \geq 0$ we have

$$0 \leq |\hat{u}| \leq (1 + |t|^2)^{s_1/2}|\hat{u}| \leq (1 + |t|^2)^{s_2/2}|\hat{u}|.$$

Therefore $u \in H^{s_2}(\mathbb{R}^N)$ implies that $u \in H^{s_1}(\mathbb{R}^N)$ and $u \in L^2(\mathbb{R}^N)$.

(b) Show that $H^s(\mathbb{R}^N) \subseteq C_0(\mathbb{R}^N)$ if $s > N/2$ (*Sobolev embedding theorem*).
Hint: Show that $\hat{u} \in L^1(\mathbb{R}^N)$ and then use Question 2.

Solution: Suppose that $u \in H^s(\mathbb{R}^N)$. We want to show that $\hat{u} \in L^1(\mathbb{R}^N)$ if $s > N/2$. Then by the inversion formula from Question 2 we conclude that $u \in C_0(\mathbb{R}^N)$. By definition

$$v := (1 + |t|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^N).$$

Applying the Cauchy-Schwarz inequality

$$\|\hat{u}\|_1 = \int_{\mathbb{R}^N} \frac{|v(t)|}{(1+|t|^2)^{s/2}} \leq \|v\|_2 \Big(\int_{\mathbb{R}^N} \frac{1}{(1+|t|^2)^s}\, dt \Big)^2.$$

Hence we need to show that the last of the above integrals is finite. Note that

$$\int_{\mathbb{R}^N} \frac{1}{(1+|t|^2)^s}\, dt = \int_{|t|\leq 1} \frac{1}{(1+|t|^2)^s}\, dt + \int_{|t|>1} \frac{1}{(1+|t|^2)^s}\, dt < \infty.$$

The first of the integrals on the right hand side is finite for all values of $s \geq 0$. Using integration in spherical coordinates we have

$$\int_{|t|>1} \frac{1}{(1+|t|^2)^s}\, dt \leq \int_{|t|>1} |t|^{-2s}\, dt = \omega_N \int_1^\infty r^{-2s} r^{N-1}\, dr = \omega_N \int_1^\infty r^{N-2s-1}\, dr,$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. The latter integral is finite if and only if $s > N/2$. This completes the proof.

(c) Show that $H^s(\mathbb{R}^N) \subseteq C_0^k(\mathbb{R}^N)$ if $s > k + N/2$ (*general Sobolev embedding theorem*). (This is based on the principle that if the Fourier transform decays fast, then the original function was quite regular and vice versa.)

Solution: We know that $\widehat{\partial u/\partial x_k}(t) = 2\pi t_k \hat{u}(t)$. Hence if we can show that $2\pi t_k \hat{u} \in L^1(\mathbb{R}^N)$, then the Fourier inversion theorem from Question 2 shows that $\partial u/\partial x_k \in C_0(\mathbb{R}^N)$. We can generalise this idea and look at k-th order partial derivatives. To show that a k-th order partial derivative is in $C_0(\mathbb{R}^N)$ it is therefore sufficient to show that $|t|^k \hat{u} \in L^1(\mathbb{R}^N)$. We proceed as in the previous part and assume that $u \in H^s(\mathbb{R}^N)$, so that

$$v := (1 + |t|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^N).$$

As in the previous part we apply the Cauchy-Schwarz inequality to get

$$\int_{\mathbb{R}^N} |t|^k |\hat{u}(t)|\, dt = \int_{\mathbb{R}^N} \frac{|v(t)||t|^k}{(1+|t|^2)^{s/2}}\, dt \leq \|v\|_2 \Big(\int_{\mathbb{R}^N} \frac{|t|^{2k}}{(1+|t|^2)^s}\, dt \Big).$$

Note that

$$\int_{\mathbb{R}^N} \frac{|t|^{2k}}{(1+|t|^2)^s}\, dt = \int_{|t|\leq 1} \frac{|t|^{2k}}{(1+|t|^2)^s}\, dt + \int_{|t|>1} \frac{|t|^{2k}}{(1+|t|^2)^s}\, dt < \infty.$$

The first of the integrals on the right hand side is finite for all values of $s \geq 0$. Using integration in spherical coordinates we have

$$\int_{|t|>1} \frac{|t|^{2k}}{(1+|t|^2)^s}\, dt \leq \int_{|t|>1} |t|^{2k-2s}\, dt = \omega_N \int_1^\infty r^{2k-2s} r^{N-1}\, dr = \omega_N \int_1^\infty r^{N+2k-2s-1}\, dr,$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. The latter integral is finite if and only if $s > k + N/2$. This completes the proof.

(d) Compute the symbol of the Laplace operator and also $u - \Delta u$.

Solution: The symbol of $-\Delta$ is

$$-\sum_{k=1}^{N}(2\pi i t_k)^2 = 4\pi\sum_{k=1}^{N} t_k^2 = 4\pi|t|^2.$$

Hence the symbol of the differential operator given by $u - \Delta u$ is

$$1 + 4\pi|t|^2$$

which is essentially the polynomial we used to define the Sobolev spaces.

(e) Prove that $u \to \lambda u - \Delta u$ is a bijection from $H^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ for all $\lambda > 0$.

Solution: First we show injectivity by proving that the kernel of the linear operator $\lambda u - \Delta u$ is trivial. Hence assume that $\lambda > 0$ and

$$\lambda u - \Delta u = 0$$

with $u \in L^2(\mathbb{R}^N)$. Applying the Fourier transform we get

$$(\lambda + 4\pi|t|^2)\hat{u} = 0$$

which implies that $\hat{u} = 0$. As the Fourier transform on $L^2(\mathbb{R}^N)$ is bijective we conclude that $u = 0$ (almost everywhere). For existence let $f \in L^2(\mathbb{R}^N)$ and again apply the Fourier transform to get

$$(\lambda + 4\pi|t|^2)\hat{u} = \hat{f}$$

As $(\lambda + 4\pi|t|^2) > \lambda > 0$ we conclude that

$$\frac{\hat{f}}{\lambda + 4\pi|t|^2} \in L^2(\mathbb{R}^N)$$

Applying the inverse Fourier transform we get $u \in L^2(\mathbb{R}^N)$. Moreover we see that

$$(\lambda + 4\pi|t|^2)\hat{u} = \hat{f} \in L^2(\mathbb{R}^N).$$

As

$$\min\{\lambda, 4\pi\}(1 + |t|^2) \le \lambda + 4\pi|t|^2 \le \max\{\lambda, 4\pi\}(1 + |t|^2)$$

we conclude that $u \in H^2(\mathbb{R}^N)$.

(f) Prove that $\to \lambda u - \Delta u$ is a bijection from $H^{s+2}(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$ for all $\lambda > 0$, that is, the partial differential equation $\lambda u - \Delta u = f$ has a unique solution $u \in H^{2+s}(\mathbb{R}^N)$ for all $f \in H^s(\mathbb{R}^N)$.

Solution: Let $f \in H^s(\mathbb{R}^N)$ and $\lambda > 0$. As $H^s(\mathbb{R}^N) \subseteq L^2(\mathbb{R}^N)$ the previous part implies the existence of a unique solution $u \in H^2(\mathbb{R}^N)$ to $\lambda u - \Delta u = f$. Taking the Fourier transform and using that $f \in H^s(\mathbb{R}^N)$ we get

$$(\lambda + 4\pi|t|^2)^{s/2}(\lambda + 4\pi|t|^2)\hat{u}(\lambda + 4\pi|t|^2)^{(s+2)/2}\hat{u} = (\lambda + 4\pi|t|^2)^{s/2}\hat{f} \in L^2(\mathbb{R}^N).$$

Hence by definition $u \in H^{s+2}(\mathbb{R}^N)$.

(g) Let $f \in C_c^\infty(\mathbb{R}^N)$. Show that $\lambda u - \Delta u = f$ has a unique solution in $C_0^\infty(\mathbb{R}^N)$ for all $\lambda > 0$.

Solution: Clearly $C_c^\infty(\mathbb{R}^N) \subseteq H^s(\mathbb{R}^N)$ for all $s \ge 0$. Hence by the previous part the solution u of $\lambda u - \Delta u = f$ lies in $H^s(\mathbb{R}^N)$ for all $s \ge 0$. By the Sobolev embedding theorem from part (c) implies that $u \in C_0^k(\mathbb{R}^N)$ for all $k \in \mathbb{N}$ and so $u \in C_0^\infty(\mathbb{R}^N)$

Remark The above is a typical way to get solutions to partial differential equations and to prove their regularity. We start off with solutions in L^2 and through the embedding theorems we get very smooth solutions if the right hand side of the equation is smooth.

Challenge questions (optional)

6. Consider the space

$$\mathcal{S}(\mathbb{R}) := \{x \in C^\infty(\mathbb{R})\colon |x|^k |f^{(n)}(x)| \to 0 \text{ as } |x| \to \infty \text{ for all } k, n \in \mathbb{N}\}.$$

This is called the *Schwartz class* of smooth very fast decaying functions.

(a) Show that $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Solution: Clearly $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ since $\operatorname{supp}(f^{(n)}) \subseteq \operatorname{supp}(f)$ is compact. Let now $f \in \mathcal{S}(\mathbb{R})$. Then there exists $R_1 > 0$ such that $|f(x)| < 1$ for all $|x| \geq R_1$. Similarly there exists $R_2 > 0$ such that $|x|^2|f(x)| \leq 1$ for all $|x| \geq R_2$. Let $R := \max\{R_1, R_2\}$. Then $(1 + |x|^2)|f(x)| \leq 2$ for $|x| > R$. Since $(1 + |x|^2)f$ is continuous it has a maximum on the compact set $\overline{B(0, R}$. Hence there exists $C_0 > 0$ such that $(1 + |x|^2)|f(x)| \leq C_0$ for all $x \in \mathbb{R}$. Therefore

$$|f(x)| \leq \frac{C_0}{1 + |x|^2}$$

for all $x \in \mathbb{R}$. Since the right hand side is integrable it follows that $f \in L^1 \cap L^\infty$ as claimed.

(b) If $f \in \mathcal{S}(\mathbb{R})$, show that $\hat{f} \in \mathcal{S}(\mathbb{R})$.

Solution: Let $f \in \mathcal{S}(\mathbb{R})$ and $k, n \in \mathbb{N}$. We show that $x^k f^{(n)}$ is integrable. There exists $R_1 > 0$ such that $|x^k f^{(n)}(x)| < 1$ for all $|x| \geq R_1$. Similarly there exists $R_2 > 0$ such that $|x|^{2+k}|f^{(n)}(x)| \leq 1$ for all $|x| \geq R_2$. Let $R := \max\{R_1, R_2\}$. Then $(1+|x|^2)|x^k f^{(n)}(x)| \leq 2$ for $|x| > R$. Since $(1 + |x|^2)x^k f^{(n)}$ is continuous it has a maximum on the compact set $\overline{B(0, R}$. Hence there exists $C > 0$ such that $(1 + |x|^2)|x^k f^{(n)}(x)| \leq C$ for all $x \in \mathbb{R}$. Therefore

$$|x^k f^{(n)}(x)| \leq \frac{C_n}{1 + |x|^2}$$

for all $x \in \mathbb{R}$. Hence $x^k f^{(n)}$ is integrable for every $n, k \in \mathbb{N}$. In particular $\hat{f} \in C^\infty(\mathbb{R})$ by Tutorial 9, Question **4**. Moreover,

$$\frac{d}{dx^n} x^k f(x)$$

is a linear combination of terms of the form $x^j f^{(m)}$ and hence by the above in $L^1(\mathbb{R})$. Next note that the formula in Tutorial 9, Question **5** is valid for $f \in \mathcal{S}(\mathbb{R})$. By a repeated application of that formula we get

$$\widehat{x^k f^{(n)}}(t) = (2\pi i t)^n \widehat{f^{(k)}}(t)$$

By the Riemann-Lebesgue Lemma we have

$$|t|^n |\widehat{f^{(k)}} \to 0$$

as $|t| \to \infty$. Since this is true for all $k, n \in \mathbb{N}$ it follows that $f \in \mathcal{S}(\mathbb{R})$.

Remark: The above makes use of the two principles:

$$\frac{d}{dx^n}\hat{f}(t) = (2\pi i)^n \widehat{x^n f}(t)$$

$$\widehat{\frac{df}{dx^n}}(t) = (2\pi i t)^n \hat{f}(t)$$

The first tells us that $\hat{f}$ is smooth if f decays very fast, and the second tells us that f is smooth if $\hat{f}$ decays fast. The fact that the Fourier transform maps $\mathcal{S}$ into intself tells us that the functions in the Schwartz space are sufficiently smooth and fast decaying so that the same is true for their Fourier transform.

(c) Show that the Fourier transform $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ is a bijection.

Solution: Let $f \in \mathcal{S}(\mathbb{R})$ and set $g(x) := \hat{f}(-x)$. By the inversion formula on $L^1(\mathbb{R})$ from Question 2 and the continuity of the functions involved

$$f(x) = \int_{\mathbb{R}} \hat{f}(t) e^{2\pi i x t} \, dt = \int_{\mathbb{R}} \hat{f}(-t) e^{-2\pi i x t} \, dt = \hat{g}(x)$$

for all $x \in \mathbb{R}$. Hence the Fourier transform is injective and surjective on $\mathcal{S}(\mathbb{R})$.