

**Tutorial 11 (Week 12)**

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MATH3969: Measure Theory and Fourier Analysis (Advanced)

Semester 2, 2011

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Web Page: <http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/>

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**Material covered**

- (1) The Radon-Nikodym theorem
- (2) An application of the projection theorem

**Outcomes**

After completing this tutorial you should

- (1) to apply the Radon-Nikodym theorem and associated properties of the density functions.
- (2) be able to apply the projection theorem in a simple situation.

**Questions to complete during the tutorial**

No tutorial because of the Quiz.

**Extra questions for further practice**

1. Let  $\mu, \nu$  and  $\lambda$  be  $\sigma$ -finite measures defined on the same  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $X$ . The Radon-Nikodym theorem asserts that if  $\nu \ll \mu$  then there exists a measurable function  $g: X \rightarrow [0, \infty)$  such that

$$\int_X f d\nu = \int_X fg d\mu$$

for all  $f \in L^1(X, \nu)$ . The function  $g$  is often called the *Radon-Nikodym derivative* and denoted  $\frac{d\nu}{d\mu}$ . We show that this is justified by proving familiar properties of derivatives.

- (a) If  $\nu \ll \mu$ , then the function  $g$  from the Radon-Nikodym theorem is essentially unique, that is, any two differ at most on a set of measure zero.
- (b) Suppose that  $\nu \ll \lambda$  and  $\mu \ll \lambda$ . Prove that

$$\frac{d(\nu + \mu)}{d\lambda} = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda}$$

almost everywhere.

- (c) Suppose  $\nu \ll \mu$  and  $\mu \ll \lambda$ . Prove the “chain rule”

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}$$

almost everywhere.

- (d) Suppose  $\nu \ll \mu$  and  $\mu \ll \nu$ . Prove that

$$\frac{d\nu}{d\mu} = \left[ \frac{d\mu}{d\nu} \right]^{-1}$$

almost everywhere.

2. Let  $\mu$  be a probability measure defined on the  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $X$ . Suppose that  $A \in \mathcal{A}$  is such that  $\mu(A) \in (0, 1)$ . Let  $f \in L_2(X)$  and define

$$g := \frac{\int_A f d\mu}{\mu(A)} 1_A + \frac{\int_{A^c} f d\mu}{\mu(A^c)} 1_{A^c}.$$

Let  $M$  be the subspace of  $L_2(X)$  spanned by  $1_A$  and  $1_{A^c}$ . Show that  $g$  is the orthogonal projection of  $f$  onto  $M$ .