The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 11 (Week 12)

MATH3969: Measure Theory and Fourier Analysis (Advanced) — Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
Lecturer: Daniel Daners

Material covered

(1) The Radon-Nikodym theorem

(2) An application of the projection theorem

Outcomes

After completing this tutorial you should

(1) to apply the Radon-Nikodym theorem and associated properties of the density functions.

(2) be able to apply the projection theorem in a simple situation.

Questions to complete during the tutorial

No tutorial because of the Quiz.

Extra questions for further practice

1. Let μ, ν and λ be σ-finite measures defined on the same σ-algebra $\mathcal{A}$ of subsets of X. The Radon-Nikodym theorem asserts that if $\nu \ll \mu$ then there exists a measurable function $g\colon X \to [0,\infty)$ such that

$$\int_X f\,d\nu = \int_X fg\,d\mu$$

for all $f \in L^1(X,\nu)$. The function g is often called the *Radon-Nikodym derivative* and denoted $\dfrac{d\nu}{d\mu}$. We show that this is justified by proving familiar properties of derivatives.

(a) If $\nu \ll \mu$, then the function g from the Radon-Nikodym theorem is essentially unique, that is, any two differ at most on a set of measure zero.

Solution: Suppose that g_1, g_2 are two functions so that

$$\nu(A) = \int_A g_1\,d\mu = \int_A g_2\,d\mu$$

for all $A \in \mathcal{A}$. Hence

$$\int_A g_1 - g_2\,d\mu = 0$$

for all $A \in \mathcal{A}$. If $h := g_1 - g_2 \neq 0$ on a set of positive measure, then $P := \{x \in X\colon h(x) > 0\}$ or $N := \{x \in X\colon h(x) < 0\}$ has positive measure. If $\mu(P) > 0$, then

$$\int_P (g_1 - g_2)\,d\mu > 0$$

which is impossible. Hence $\mu(P) = 0$ and similarly $\mu(N) = 0$, so $g_1 = g_2$ almost everywhere.

(b) Suppose that $\nu \ll \lambda$ and $\mu \ll \lambda$. Prove that

$$\frac{d(\nu+\mu)}{d\lambda} = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda}$$

almost everywhere.

Solution: By the Radon-Nikodym theorem there exist $g_1 = \frac{d\nu}{d\lambda}$ and $g_2 = \frac{d\mu}{d\lambda}$ such that

$$\nu(A) = \int_A g_1 \, d\lambda \qquad \text{and} \qquad \mu(A) = \int_A g_2 \, d\lambda$$

for all $A \in \mathcal{A}$. Hence

$$(\nu+\mu)(A) = \nu(A) + \mu(A) = \int_A (g_1 + g_2) \, d\lambda$$

for all $A \in \mathcal{A}$. By uniqueness of the Radon-Nikodym derivative we have

$$\frac{d(\nu+\mu)}{d\lambda} = g_1 + g_2 = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda}$$

almost everywhere.

(c) Suppose $\nu \ll \mu$ and $\mu \ll \lambda$. Prove the "chain rule"

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

almost everywhere.

Solution: By the Radon-Nikodym theorem there exist functions $g_1, g_2 \colon X \to [0,\infty)$ such that

$$\int_X f \, d\nu = \int_X f g_1 \, d\mu \qquad \text{and} \qquad \int_X f \, d\mu = \int_X f g_2 \, d\lambda$$

for all measurable functions $f \colon X \to [0,\infty)$. Hence in particular

$$\nu(A) \int_X 1_A \, d\nu = \int_X 1_A g_1 \, d\mu = \int_X 1_A g_1 g_2 \, d\lambda = \int_A g_1 g_2 \, d\lambda$$

for all $A \in \mathcal{A}$. By uniqueness almost everywhere of the Radon-Nikodym derivative we have

$$\frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} g_1 g_2 = \frac{d\nu}{d\lambda}$$

as claimed.

(d) Suppose $\nu \ll \mu$ and $\mu \ll \nu$. Prove that

$$\frac{d\nu}{d\mu} = \left[\frac{d\mu}{d\nu}\right]^{-1}$$

almost everywhere.

Solution: By the Radon-Nikodym theorem there exists a function $g \colon X \to [0,\infty)$ such that

$$\int_X f \, d\nu = \int_X f g \, d\mu$$

for all measurable functions $f \colon X \to [0,\infty)$. If we apply the above identity to f replaced by the non-negative measurable function f/g we get

$$\int_X \frac{f}{g} \, d\nu = \int_X f \, d\mu$$

for all measurable functions $f \colon X \to [0,\infty)$. By the uniqueness of the Radon-Nikodym derivative

$$\frac{d\nu}{d\mu} = g = \frac{1}{1/g} = \left[\frac{d\mu}{d\nu}\right]^{-1}$$

2. Let μ be a probability measure defined on the σ-algebra $\mathcal{A}$ of subsets of X. Suppose that $A \in \mathcal{A}$ is such that $\mu(A) \in (0,1)$. Let $f \in L_2(X)$ and define

$$g := \frac{\int_A f \, d\mu}{\mu(A)} 1_A + \frac{\int_{A^c} f \, d\mu}{\mu(A^c)} 1_{A^c}.$$

Let M be the subspace of $L_2(X)$ spanned by 1_A and 1_{A^c}. Show that g is the orthogonal projection of f onto M.

Solution: We only need to show that $(f - g \mid 1_A) = (f - g \mid 1_{A^c}) = 0$. Then $(f - g \mid h) = 0$ for all linear combinations of 1_A and 1_{A^c}. By definition of the L^2-inner product

$$\begin{aligned}(f - g \mid 1_A) = \int_X (f-g) 1_A \, d\mu = \int_A (f-g) \, d\mu = \int_A f - \frac{\int_A f \, d\mu}{\mu(A)} 1_A + \frac{\int_{A^c} f \, d\mu}{\mu(A^c)} 1_{A^c} \, d\mu \\ = \int_A f \, d\mu - \int_A \frac{\int_A f \, d\mu}{\mu(A)} \, d\mu = \int_A f \, d\mu - \frac{\int_A f \, d\mu}{\mu(A)} \mu(A) = 0.\end{aligned}$$

Similarly $(f - g \mid 1_{A^c}) = 0$, completing the proof.