The University of Sydney
School of Mathematics and Statistics

# Tutorial 12 (Week 13)

MATH3969: Measure Theory and Fourier Analysis (Advanced) Semester 2, 2011

Web Page: http://www.maths.usyd.edu.au/u/UG/SM/MATH3969/
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## Material covered

(1) applications of measure theory to probability theory

(2) conditional expectation

## Outcomes

After completing this tutorial you should

(1) have an idea on how measure theory is applied to probability theory.

(2) be familiar with conditional expectation.

## Questions to complete during the tutorial

**1.** Let $\Omega = [0,1]$ and $P = m$ the Lebesgue measure. Then $[0,1]$ is a probability space. Give examples of two distinct random variables which have the same distribution.

**2.** Let $X\colon \Omega \to \mathbb{R}$ be a random variable on the probability space $(\Omega, \mathcal{A}, P)$.

(a) Prove that

$$P[|X| \geq \alpha] \leq \frac{1}{\alpha^p} \int_{[|X|\geq\alpha]} |X|^p \, dP \leq \frac{1}{\alpha^p} \mathbb{E}[|X|^p]$$

for all $\alpha > 0$ and $1 \leq p < \infty$.

(b) Prove *Chebychev's inequality*

$$P[|X - \mu| \geq \alpha] \leq \frac{1}{\alpha^2} \operatorname{Var}(X)$$

for all $\alpha > 0$, where $\mu := \mathbb{E}[X]$ is the expectation of $X$.

**3.** Let $\Omega$ be the space obtained by coin tossing countably many times. Such coin tosses can be represented as infinite sequences of the form $\Omega := \{(a_1, a_2, \dots)\colon a_k = 0 \text{ or } 1\}$. A zero means head and a one means tail for instance. Such sequences can be interpreted as binary expansions of the number $\alpha = \sum_{k=1}^{\infty} \frac{a_k}{2^k}$ which is between zero and one. With that identification we can set $\Omega = [0,1)$.

(a) Denote by $A_k := \{(a_1, \dots, a_k, \dots) \in \Omega\colon a_k = 1\}$. Using that $\Omega = [0,1)$ sketch $A_1$, $A_2$, $A_3$ and then describe the sets $A_k$ for general $k$. What is the probability of $A_k$, and how does it compare to the Lebesgue measure of $A_k$?

(b) Show that every interval of the form $I_{n,j} = [j/2^n, (j+1)/2^n)$ $(j = 0, \dots, 2^{n-1})$ can be written as a finite intersection of the sets $A_k$ and their complements.

(c) Argue why the probability measure in the above situation is the Lebesgue measure on $[0,1)$.

## Extra questions for further practice

**4.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{A}_0$ a $\sigma$-algebra with $\mathcal{A}_0 \subseteq \mathcal{A}$. Then clearly $L^2(\Omega, \mathcal{A}_0, P)$ is a closed subspace of $L^2(\Omega, \mathcal{A}, P)$ and therefore, by the projection theorem in a Hilbert space, for every random variable $X \in L^2(\Omega, \mathcal{A}, P)$ there exists $X_0 \in L^2(\Omega, \mathcal{A}_0, P)$ such that $X - X_0$ is orthogonal to $L^2(\Omega, \mathcal{A}_0, P)$. Prove that $X_0 = \mathbb{E}[X|\mathcal{A}_0]$ almost everywhere.

The following question generalises *Jensen's inequality* to conditional expectation.

**5.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{A}_0$ a $\sigma$-algebra with $\mathcal{A}_0 \subseteq \mathcal{A}$. Let $X \in L^1(\Omega, \mathcal{A}, P)$ be a random variable and $X_0 := \mathbb{E}[X|\mathcal{A}_0]$. Finally let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a convex function.

(a) Show that there exists an increasing function $m\colon \mathbb{R} \to \mathbb{R}$ such that

$$m(t) = \sup_{s<t} \frac{\varphi(s) - \varphi(t)}{s - t}$$

for all $s, t \in \mathbb{R}$.

(b) Let $X \in L^\infty(\Omega, \mathcal{A}, P)$. Prove that $|X_0| \leq |X|$ almost everywhere.

(c) Let $X \in L^\infty(\Omega, \mathcal{A}, P)$. Prove that

$$\mathbb{E}[\varphi \circ X|\mathcal{A}_0] \geq \varphi \circ X_0 = \varphi \circ \mathbb{E}[X|\mathcal{A}_0]$$

almost everywhere.
*Hint:* Use (**??**) to show that $\varphi \circ X \geq \varphi \circ X_0 - (m \circ X_0)(X - X_0)$, and then integrate. Check the measurability of the functions carefully.

(d) If $\varphi \circ X \in L^1(\Omega, \mathcal{A}, P)$, prove that $\varphi \circ \mathbb{E}[X|\mathcal{A}_0] \leq \mathbb{E}[\varphi \circ X|\mathcal{A}_0]$ almost everywhere. (This generalises Jensen's inequality.)
*Hint:* Approximate $X$ by a sequence of bounded functions.

**6.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{A}_0$ a $\sigma$-algebra with $\mathcal{A}_0 \subseteq \mathcal{A}$. Let $X \in L^1(\Omega, \mathcal{A}, P)$ be a random variable. Use Question **??** to show that the linear map $X \mapsto \mathbb{E}[X|\mathcal{A}_0]$ is continuous from $L^p(\Omega, \mathcal{A}, P)$ to $L^p(\Omega, \mathcal{A}_0, P)$ if $1 \leq p < \infty$.

The question below is an application of the theory of Fourier transforms to probability.

**7.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X\colon \Omega \to \mathbb{R}$ a random variable. Let $P_X$ be the distribution of $X$, that is, $P_X[A] := P[X \in A]$ for every Borel set $A \subseteq \mathbb{R}$. Define

$$\varphi_X(t) := \int_{\mathbb{R}} e^{ist}\, dP_X(s)$$

Then $\varphi_X\colon \mathbb{R} \to \mathbb{C}$ is called the *characteristic function* of the random variable $X$. Note that up to some normalising factors this is the inverse Fourier transform of the measure $P_X$.

(a) Show that $\varphi_X$ is continuous and that $\|\varphi_X\|_\infty \leq 1$.

(b) Show that $\varphi_X(t) = \mathbb{E}[e^{itX}]$

(c) Suppose that $P_X$ has a density function. Show that $\varphi_X \in C_0(\mathbb{R})$.

(d) Suppose that $P_X = \delta_0$, where $\delta_0$ is the Dirac measure concentrated at $t = 0$. Compute the characteristic function $\varphi_X$.

(e) Compute the characteristic function of a normally distributed random variable, that is, a random variable with distribution density $(2\pi\sigma)^{-1/2}e^{-|x-\mu|^2/2\sigma}$ (mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$).