The University of Sydney

FACULTIES OF ARTS, ECONOMICS, EDUCATION,
ENGINEERING AND SCIENCE

MATH3974

Paper : Fluid Dynamics

November, 2009 | Time allowed: Two hours

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THIS PAPER IS CONFIDENTIAL

INSTRUCTIONS

(*i*) A list of vector identities is included at the end of the paper.
(*ii*) Non-programmable calculators may be used.
(*iii*) All questions are worth equal marks.
(*iv*) All questions may be attempted.

You may assume any formulae derived in lectures without proof, except in question 4. You may also freely use without proof the vector identities and curvilinear coordinate formulae given at the end of the paper.

1. (*i*) Write down or derive the velocity components and velocity potential for the following ideal (ie inviscid, incompressible, irrotational) steady flows:

(a) A point source of strength S at the origin

(b) A three-dimensional dipole of strength M at the origin with its axis aligned along the $z-$axis (use spherical polar coordinates)

(c) A similar dipole moved to the point $(0, 0, h)$ in Cartesian coordinates (hint: take the potential from the last part and convert it to Cartesian coordinates, then move the origin to $z = h$. Give the velocity components in the Cartesian system).

(*ii*) A semi-infinite domain of ideal fluid occupies the upper half-space $z > 0$; $z = 0$ is a rigid impermeable boundary. A three-dimensional axisymmetric dipole with strength M is situated at the point $(0, 0, d)$ when expressed in Cartesian coordinates. The axis of the dipole is aligned parallel to the $z-$axis. Far from the origin the flow tends to zero sufficiently fast for the uniqueness theorem to apply, and the pressure is zero. Use an image method to write down the velocity potential for the resulting flow, and take its gradient to find the velocity components. Use Bernoulli to calculate the

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pressure distribution on the plane $z = 0$, ignoring gravity. Hence show that the total pressure force acting on the plane is $-3\rho M^2/(32\pi d^4)$, so that the dipole is attracted towards the wall.

2. A flow with circular streamlines has an axisymmetric pressure distribution and velocity $\mathbf{u} = v(R,t)\mathbf{e}_\phi$, referred to cylindrical polar coordinates (R, ϕ, z). Show that the ϕ-component of the Navier-Stokes equation is

$$\frac{\partial v}{\partial t} = \nu\Big(\frac{\partial^2 v}{\partial R^2} + \frac{1}{R}\frac{\partial v}{\partial R} - \frac{v}{R^2}\Big)\,.$$

Use this equation to investigate how an initial line vortex singularity spreads due to viscous diffusion. Defining the circulation $\kappa(R,T)$ at radius R as $2\pi R v(R,t)$, reexpress the above equation in terms of κ, showing

$$\frac{\partial \kappa}{\partial t} = \nu\Big(\frac{\partial^2 \kappa}{\partial R^2} - \frac{1}{R}\frac{\partial \kappa}{\partial R}\Big)\,.$$

The initial condition appropriate for a line vortex at time $t = 0$ is $\kappa(R,0) = \kappa_0$, where κ_0 is the vortex strength. At any later time, viscosity ensures the velocity $v(0,t)$ is finite; thus the above equation is to be solved with the given initial condition and the boundary conditions $\kappa = 0$ at $R = 0$ and $\kappa \to \kappa_0$ as $R \to \infty$. Seek a similarity solution $\kappa = f(\eta)$, where $\eta = R/\sqrt{4\nu t}$. Substitute into the problem just described, and show that the PDE becomes an ODE for f. Solve it subject to the specified boundary conditions, and hence derive the result

$$v(R,t) = \frac{\kappa_0}{2\pi R}\Big(1 - e^{-R^2/4\nu t}\Big)\,.$$

Deduce that when $R \gg \sqrt{4\nu t}$, the velocity is almost the same as for the original line vortex, but that when $R \ll \sqrt{4\nu t}$, the velocity is close to rigid body rotation (with an angular velocity you should specify).

3. (*i*) Given a complex potential $w(z) = \Phi(x,y) + i\psi(x,y)$, use the Cauchy-Riemann equations to show that the curves Φ =constant and ψ = constant intersect at right angles. Show also that the curves Φ =constant give the streamlines of some other 2-D flow, and identify each flow in the three cases where $w(z) = Uz$, $w(z) = S\log z$ and $w(z) = M/z$, U, S and M being real constants.

(*ii*) Two line vortices each of circulation κ are situated on the real axis at $z = \pm a$. A third vortex of strength $-\kappa/2$ is at the origin. Write down the complex potential for this problem and use it to find the complex velocity $u - iv$. Hence show that each vortex is at rest, find the locations of any other stagnation points, and sketch the streamlines of the flow.

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4. In this question, we establish Rayleigh's inflection point criterion for the instability of 2-D shear flows.

(*i*) In the two-dimensional channel $-\infty < x < \infty$, $-L \le y \le L$, consider the shear flow given by $u(x, y, t) = U(y)$ and $v(x, y, t) = 0$, where $U(y)$ is a differentiable function of y. Show that this velocity, along with the pressure field $p(x, y, t) = P_0$ (a constant), is a solution to the inviscid two-dimensional incompressible Navier-Stokes equations with constant density ρ and no body forces.

(*ii*) Consider now a perturbation given by

$$\mathbf{u}(x, y, t) = (U(y) + u_1(x, y, t), 0 + v_1(x, y, t)) \text{ and}$$
$$p(x, y, t) = P_0 + p_1(x, y, t)\,,$$

where quantities subscripted with 1 are considered small. By substituting in the appropriate dynamical equations mentioned above and linearising, establish that

$$\frac{\partial u_1}{\partial t} + U(y)\,\frac{\partial u_1}{\partial x} + v_1\,\frac{dU}{dy} = -\frac{1}{\rho}\,\frac{\partial p_1}{\partial x}\,,$$
$$\frac{\partial v_1}{\partial t} + U(y)\,\frac{\partial v_1}{\partial x} = -\frac{1}{\rho}\,\frac{\partial p_1}{\partial y}\,,$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0\,.$$

(*iii*) Assume that

$$u_1 = \text{Re}\left\{\hat{u}(y)\,e^{i(kx-\omega t)}\right\}$$
$$v_1 = \text{Re}\left\{\hat{v}(y)\,e^{i(kx-\omega t)}\right\}$$
$$p_1 = \text{Re}\left\{\hat{p}(y)\,e^{i(kx-\omega t)}\right\}$$

where the wavenumber k is positive. By substituting into the above equations and eliminating $\hat{u}$ and $\hat{p}$, establish Rayleigh's equation

$$\hat{v}'' + \left(\frac{k\,U''(y)}{\omega - k\,U(y)} - k^2\right)\,\hat{v} = 0\,.$$

and argue that the relevant boundary conditions are $\hat{v}\,(\pm L) = 0$.

(*iv*) Multiply Rayleigh's equation by $\hat{v}^*$ (the complex conjugate of $\hat{v}$) and integrate from $-L$ to L. Write $\omega = \omega_r + i\,\omega_i$, and by separating out real and imaginary parts, argue that a necessary condition for linear instability is that $U''(y)$ must change sign in the flow.

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Appendix-Vector Identities and Curvilinear Coordinates.

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$(\mathbf{a}\cdot\nabla)\mathbf{a} = \nabla(a^2/2) - \mathbf{a}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\phi\mathbf{a}) = \phi\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\phi$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b}$$

$$\nabla\times(\phi\mathbf{a}) = \phi\nabla\times\mathbf{a} + \nabla\phi\times\mathbf{a}$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot\mathbf{a}$$

General curvilinear coordinates q_1, q_2, q_3

Let $\mathbf{e}_i$ be a unit vector along the q_i-axis, with

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \, .$$

$$\text{Then} \quad \nabla\Phi = \sum_i \frac{1}{h_i}\frac{\partial\Phi}{\partial q_i}\mathbf{e}_i$$

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1 h_2 h_3}\sum_{i,j,k}\frac{\partial}{\partial q_i}(h_j h_k F_i)$$

(where here and subsequently the sum is over i with j and k selected cyclically, so that for $i = 1$ j and k are 2 and 3 repectively, for $i = 2$ they are 3 and 1, for $i = 3$ they are 1 and 2)

$$\nabla\times\mathbf{F} = \sum_{i,j,k}\frac{1}{h_j h_k}\Big[\frac{\partial}{\partial q_j}(h_k F_k) - \frac{\partial}{\partial q_k}(h_j F_j)\Big]\mathbf{e}_i$$

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\sum_{i,j,k}\frac{\partial}{\partial q_i}\Big(\frac{h_j h_k}{h_i}\frac{\partial\Phi}{\partial q_i}\Big)$$

$$\nabla^2\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F})$$

$$(\mathbf{B}\cdot\nabla)\mathbf{A} = \sum_{i,j,k}\Big\{\mathbf{B}\cdot\nabla A_i + \frac{A_j}{h_i h_j}\Big(B_i\frac{\partial h_i}{\partial q_j} - B_j\frac{\partial h_j}{\partial q_i}\Big) + \frac{A_k}{h_i h_k}\Big(B_i\frac{\partial h_i}{\partial q_k} - B_k\frac{\partial h_k}{\partial q_i}\Big)\Big\}\mathbf{e}_i$$

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Cylindrical polar coordinates (R, ϕ, z)

$$\nabla\Phi = \frac{\partial\Phi}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z$$

$$\nabla\cdot\mathbf{F} = \frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{1}{R}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla\times\mathbf{F} = \Big[\frac{1}{R}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\Big]\mathbf{e}_R + \Big[\frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R}\Big]\mathbf{e}_\phi + \frac{1}{R}\Big[\frac{\partial}{\partial R}(RF_\phi) - \frac{\partial F_R}{\partial\phi}\Big]\mathbf{e}_z$$

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\Big(R\frac{\partial\Phi}{\partial R}\Big) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\begin{aligned}\nabla^2\mathbf{F} =& \Big[\nabla^2 F_R - \frac{1}{R^2}F_R - \frac{2}{R^2}\frac{\partial F_\phi}{\partial\phi}\Big]\mathbf{e}_R \\ &+ \Big[\nabla^2 F_\phi - \frac{1}{R^2}F_\phi + \frac{2}{R^2}\frac{\partial F_R}{\partial\phi}\Big]\mathbf{e}_\phi + \nabla^2 F_z\,\mathbf{e}_z\end{aligned}$$

$$\begin{aligned}(\mathbf{B}\cdot\nabla)\mathbf{A} =& \big[\mathbf{B}\cdot\nabla A_R - B_\phi A_\phi/R\big]\mathbf{e}_R \\ &+ \big[\mathbf{B}\cdot\nabla A_\phi + B_\phi A_R/R\big]\mathbf{e}_\phi + \mathbf{B}\cdot\nabla A_z\,\mathbf{e}_z\end{aligned}$$

Spherical polar coordinates (r, θ, ϕ)

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi$$

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2 F_r\Big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta F_\theta\Big) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\begin{aligned}\nabla\times\mathbf{F} =& \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\Big]\mathbf{e}_r \\ &+ \frac{1}{r}\Big[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\Big]\mathbf{e}_\theta + \frac{1}{r}\Big[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\Big]\mathbf{e}_\phi\end{aligned}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Phi}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$\begin{aligned}\nabla^2\mathbf{F} =& \Big[\nabla^2 F_r - \frac{2}{r^2}F_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta F_\theta\Big) - \frac{2}{r^2\sin\theta}\frac{\partial F_\phi}{\partial\phi}\Big]\mathbf{e}_r \\ &+ \Big[\nabla^2 F_\theta - \frac{1}{r^2\sin^2\theta}F_\theta + \frac{2}{r^2}\frac{\partial F_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\phi}{\partial\phi}\Big]\mathbf{e}_\theta \\ &+ \Big[\nabla^2 F_\phi - \frac{1}{r^2\sin^2\theta}F_\phi + \frac{2}{r^2\sin^2\theta}\frac{\partial F_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial F_\theta}{\partial\phi}\Big]\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}(\mathbf{B}\cdot\nabla)\mathbf{A} =& \big[\mathbf{B}\cdot\nabla A_r - (B_\theta A_\theta + B_\phi A_\phi)/r\big]\mathbf{e}_r \\ &+ \big[\mathbf{B}\cdot\nabla A_\theta + (B_\theta A_r - \cot\theta B_\phi A_\phi)/r\big]\mathbf{e}_\theta \\ &+ \big[\mathbf{B}\cdot\nabla A_\phi + (B_\phi A_r + \cot\theta B_\phi A_\theta)/r\big]\mathbf{e}_\phi\end{aligned}$$