THE UNIVERSITY OF SYDNEY

APPLIED MATHEMATICS III

Fluid Dynamics 2011

# Assignment 3

Due: 6pm Monday June 6th

**Post your assignment into the box labelled MATH3974 opposite the Carslaw lifts on Level 6**

1. Do question 8.4 from Acheson's *Elementary Fluid Dynamics*, pp 293-295 (this is reprinted below for your convenience, but note that the book itself, which is on special reserve in the Scitech Library, has some hints on page 380 which you may find it helpful to consult).

8.4. A two-dimensional jet emerges from a narrow slit in a wall into fluid which is at rest. If the jet is thin, so that $u$ varies much more rapidly across the jet than along it (Fig. 8.16), the arguments of boundary layer

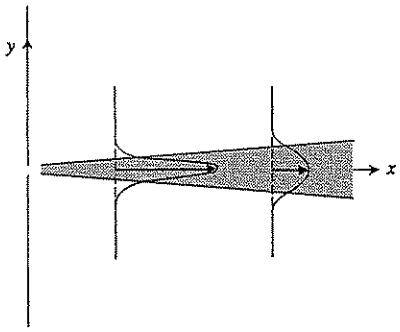


**Fig. 8.16.** A thin 2-D jet.

theory apply and

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (8.64)$$

in the jet. (The pressure gradient is zero in the jet because it is zero in the stationary fluid outside; cf. eqn (8.8).) The boundary conditions are that $u\to 0$ as we leave the jet and $\partial u/\partial y=0$ at $y=0$, as the motion is symmetrical about the $x$-axis.

By integrating eqn (8.64) across the jet, show that

$$\int_{-\infty}^{\infty} u\frac{\partial u}{\partial x}\,\mathrm{d}y+\int_{-\infty}^{\infty} v\frac{\partial u}{\partial y}\,\mathrm{d}y=\left[\nu\frac{\partial u}{\partial y}\right]_{-\infty}^{\infty}$$

and hence that

$$M=\rho\int_{-\infty}^{\infty} u^2\,\mathrm{d}y=\text{constant}. \qquad (8.65)$$

Seek a similarity solution for the stream function

$$\psi=F(x)f(\eta), \qquad \text{where } \eta=y/g(x),$$

and show first that

$$F(x)=(3M/2\rho)^{\frac{1}{2}}[g(x)]^{\frac{1}{2}} \qquad (8.66)$$

if we set

$$\int_{-\infty}^{\infty}[f'(\eta)]^2\,\mathrm{d}\eta=\tfrac{2}{3}. \qquad (8.67)$$

(The choice of $\frac{2}{3}$ is arbitrary, but it keeps the numerical factors relatively simple in what follows.) Then use the jet equation (8.64) to deduce that $g(x)$ must be proportional to $x^{\frac{2}{3}}$.

Show that the particular choice

$$g(x)=(3x\nu)^{\frac{2}{3}}(2\rho/3M)^{\frac{1}{3}}$$

leads to the problem

$$f'''+ff''+f'^2=0$$

subject to

$$f(0)=f''(0)=0, \qquad f'(\infty)=0.$$

Integrate twice to obtain

$$f'+\tfrac{1}{2}f^2=\text{constant}$$

and deduce that $f=2A\tanh(A\eta)$. Then use eqn (8.67) to determine $A$, and show that

$$u=\tfrac{1}{2}\left(\frac{3M^2}{4\nu\rho^2 x}\right)^{\frac{1}{3}}\operatorname{sech}^2(\tfrac{1}{2}\eta).$$

Give a rough estimate of the width of the jet at a distance $x$ from the slit, and deduce a condition involving $M/\rho$, $\nu$, and $x$ which must be satisfied for the above boundary-layer-type treatment to be valid.

[It is possible to investigate an axisymmetric jet emerging from a small hole in a wall in a similar manner (Goldstein 1938, pp. 147–148; Rosenhead 1963, pp. 452–455; Schlichting 1979, pp. 230–234). Using cylindrical polar coordinates, the momentum flux

$$M=2\pi\rho\int_0^{\infty} ru_z^2\,\mathrm{d}r$$

is independent of $z$, the distance from the hole, and the width of the jet is of order $z/(M/\rho\nu^2)^{\frac{1}{2}}$, so the boundary-layer-type treatment is valid only if $M/\rho\nu^2\gg 1$.

This axisymmetric jet is in fact a limiting case (for large $M/\rho\nu^2$) of an exact, jet-like solution of the full Navier–Stokes equations (Rosenhead 1963, pp. 150–155; Batchelor 1967, pp. 205–211). At the opposite extreme, $M/\rho\nu^2\ll 1$, the jet is very broad, and its Stokes stream function is, in spherical polar coordinates,

$$\Psi=\frac{M}{8\pi\mu}r\sin^2\theta,$$

which is a solution of the slow flow equations (see §7.2).

In practice, jets become unstable and turbulent at much lower Reynolds numbers than do boundary layers. Furthermore, the 'solutions' above for the jet emerging from a hole in a wall do not, of course, satisfy the no-slip condition on the wall. This matter has been put to right only quite recently, with interesting consequences for the flow outside the jet, particularly at lower Reynolds numbers (Schneider 1981, 1985; Zauner 1985, especially pp. 115 and 116).]

**2.** The Kelvin-Helmholtz instability occurs on an interface between two fluids moving relative to each other with a tangential discontinuity at the interface. Suppose the undisturbed interface is at $y = 0$, with the fluid above moving at $+V$ in the $x$-direction, and the fluid below moving at $-V$ (any such discontinuity can be brought into this form by an appropriate Galilean transformation). Assuming irrotational, incompressible flow, perturb this basic flow with an additional velocity $\mathbf{u}_u(x, y, t)$ (velocity potential $\phi_u(x, y, t)$) in the *upper* region $y > 0$, $\mathbf{u}_\ell(x, y, t)$ (velocity potential $\phi_\ell(x, y, t)$) in the $\ell$ower region $y < 0$. Seek plane wave solutions $\phi_{u,\ell} = \Phi_{u,\ell}(y)e^{i(kx-\omega t)}$ to Laplace's equation appropriate for the two regions, taking into account the boundary condition that the disturbance must decay as $y \to \pm\infty$, and giving the form of $\phi_{u,\ell}(y)$. Show using an appropriate Bernoulli equation that the linearised pressure perturbation is given by

$$p_{u,\ell} = -\rho\left(\frac{\partial \phi_{u,\ell}}{\partial t} \pm V\frac{\partial \phi_{u,\ell}}{\partial x}\right),$$

$\rho$ being the (constant) fluid density and $+$ being for the top layer, $-$ for the bottom.

Associated with the velocity disturbance is a disturbance $\zeta = \zeta_0 e^{i(kx-\omega t)}$ to the interface between the fluids. The rate of change of $\zeta$ following the interface has to be the same as the vertical velocity (evaluated in the linearised approximation at $y = 0$), for both the upper and lower fluids. This gives the boundary conditions

$$\frac{\partial \phi_u}{\partial y} = \left(\frac{D\zeta}{Dt}\right)_u = \frac{\partial \zeta}{\partial t} + V\frac{\partial \zeta}{\partial x}$$

$$\frac{\partial \phi_\ell}{\partial y} = \left(\frac{D\zeta}{Dt}\right)_\ell = \frac{\partial \zeta}{\partial t} - V\frac{\partial \zeta}{\partial x}$$

at $y = 0$. Show that these, together with the condition that the pressure is continuous at $y = 0$, give three linear homogeneous equations for $\zeta_0$ and the amplitudes of $\Phi_{u,\ell}$ at $y = 0$. Show that these only have a non-zero solution if the dispersion relation

$$\omega^2 = -V^2k^2$$

is satisfied. Deduce that an initial perturbation comprising all Fourier components will grow exponentially, so that the flow is unstable. Which Fourier components are the most unstable? Do you think this problem is well-posed? What physics that we have ignored do you think would be important for correcting the apparent paradox that has resulted?

(This is treated in textbooks; see e.g. Batchelor p.511ff, Drazin & Reid p14ff, or Drazin Ch. 3)