The circulation around a simple closed contour \( C \) is defined as

\[
\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l}
\]

where element along \( C \)

\( \mathbf{u} \cdot d\mathbf{l} \) measures "circulation"

Now consider a family of contours \( C(t) \) which consist of the same fluid particles as time proceeds, i.e., which are advected by the flow

\[ C(t') = \varphi_{t' - t} C(t) \]  \( \varphi : \) fluid flow map
Kelvin's Circulation Theorem

Let $C(t)$ be a simple closed material curve in an ideal fluid. Then, if either (i) $p$ is constant or (ii) the fluid is barotropic, the circulation $\Gamma_{C(t)}$ of $\bar{v}$ on $C$ is invariant under the flow:

$$\frac{d}{dt} \Gamma_{C(t)} = 0$$

Remarks:
* this is in general not true for any closed curve $C$ fixed in space
* the Theorem does not require the fluid region to be simply connected, i.e. the surface enclosed by $C$ is allowed to contain non-fluid material (i.e. wings).
* In the proof one needs the viscosity to be zero only on the contour $C$; so viscous effects are allowed within the fluid, away from $C$, i.e. boundary layers on airfoils.
proof:

Transport Theorem for currents:

$$\frac{d}{dt} \int_{C(t)} \hat{u} \cdot d\ell = \int_{C(t)} \frac{D\hat{u}}{Dt} \cdot d\ell$$

proof: Parameterize $C = C(\alpha)$ by $\hat{x}(s), s \in [0, 1]$

Then for $t > 0$ $C(t)$ is parameterized by $\psi_t(\hat{x}(s)) = \psi_{0t}(s)$

Then

$$\frac{d}{dt} \int_{C(t)} \hat{u} \cdot d\ell = \frac{d}{dt} \int_0^1 \hat{u}(\psi_t(\hat{x}(s), t)) \cdot \frac{\partial}{\partial s} \psi_t(\hat{x}(s)) \, ds$$

$$= \int_0^1 \frac{D\hat{u}}{Dt}(\psi_t(\hat{x}(s), t)) \cdot \frac{\partial}{\partial s} \psi_t(\hat{x}(s)) \, ds$$

$$+ \int_0^1 \hat{u}(\psi_t(\hat{x}(s), t)) \cdot \frac{\partial}{\partial s} \psi_t(\hat{x}(s)) \, ds$$

Since $\frac{\partial \psi_t}{\partial t} = \hat{u}$ we have

$$\hat{u}(\psi_t(\hat{x}(s), t)) \cdot \frac{\partial}{\partial s} \psi_t(\hat{x}(s)) = \hat{u}(\psi_t(\hat{x}(s), t)) \cdot \frac{\partial}{\partial s} \psi(\hat{x}(s)) + \frac{1}{\epsilon} \frac{\partial}{\partial s} (\hat{u} \cdot \hat{u})(\psi_t(\hat{x}(s), t))$$
\[
\Rightarrow \frac{d}{dt} \int \vec{u} \cdot d\vec{L} = \int_0^1 \frac{D\vec{u}}{Dt} (\psi(x(s), t)) \cdot \frac{\partial}{\partial s} \psi(x(s), t) \, ds \\
= \int_C \frac{D\vec{u}}{Dt} \cdot d\vec{L}
\]

Employing the Euler equations:

\[
\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla \rho - \nabla V
\]

we obtain

\[
\frac{d}{dt} \int \vec{u} \cdot d\vec{L} = \int_C -\left(\frac{1}{\rho} \nabla \rho + \nabla V\right) \cdot d\vec{L}
\]

\[
= -\int \nabla \left(\int \frac{df}{f(\vec{r})} + V\right) \cdot d\vec{L}
\]

(provided all variables are single-valued) = 0
* Generation of lift on an airfoil

at \( t = 0 \)

\[ \Gamma = \frac{\delta u \cdot dl}{c_0} \]

\( 0 < t < 1 \)

miscellaneous boundary layer

Stating vortex

\( t \gg 1 \)

\[ \text{Kelvin: } \Gamma = \frac{\delta u \cdot dl}{c_t} \]

If \( C_0 \) is far away from \( C_t \), then \( C_t \) can be viewed as a materially advected \( C_0 \).