The circulation around a simple closed contour \( \Gamma \) is defined as
\[
\Gamma = \oint_{\gamma} \mathbf{u} \cdot d\mathbf{e}
\]
where \( d\mathbf{e} \) is a line element along \( \gamma \).

Now consider a family of contours \( \mathcal{C}(t) \) which consist of the same fluid particles as time proceeds, i.e. which are advected by the flow.

\[
\mathcal{C}(t_2) = \Phi_{t_2-t_1} \mathcal{C}(t_1) \quad \Phi : \text{fluid flow map}
\]
Kelvin's Circulation Theorem

Let $C(t)$ be a simple closed material curve in an ideal fluid. Then, if either (i) $\rho$ is constant or (ii) the fluid is barotropic, the circulation $\Gamma_C$ of $\mathbf{u}$ on $C$ is invariant under the flow:

$$\frac{d}{dt} \Gamma_C = 0$$

Remarks:
* this is in general not true for any closed curve $C$ fixed in space
* the Theorem does not require the fluid region to be simply connected, i.e., the surface enclosed by $C$ is allowed to contain non-fluid material (i.e., wings).
* in the proof one needs the viscosity to be zero only on the contour $C$; so viscous effects are allowed within the fluid, away from $C$, i.e., boundary layer on airfoils.
proof: Transport Theorem for curves:

\[
\frac{d}{dt} \int_{C(t)} \mathbf{u} \cdot d\mathbf{\ell} = \int_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{\ell}
\]

Proof: Parametrize \( C = C(s) \) by \( \mathbf{x}(s), s \in [0, 1] \)

Then for \( t > 0 \) \( C(t) \) is parametrized by \( \psi_t(\mathbf{x}(s)) = \mathbf{p}(s) \)

Then

\[
\frac{d}{dt} \int_{C(t)} \mathbf{u} \cdot d\mathbf{\ell} = \frac{d}{dt} \int_0^1 \mathbf{u}(\psi_t(\mathbf{x}(s), t)) \cdot \frac{\partial}{\partial s} \psi_t(\mathbf{x}(s)) \, ds
\]

\[
= \int_0^1 \frac{D\mathbf{u}}{Dt}(\psi_t(\mathbf{x}(s), t)) \cdot \frac{\partial}{\partial s} \psi_t(\mathbf{x}(s)) \, ds
\]

\[
+ \int_0^1 \mathbf{u}(\psi_t(\mathbf{x}(s), t)) \cdot \frac{\partial}{\partial t} \frac{\partial}{\partial s} \psi_t(\mathbf{x}(s)) \, ds
\]

Since \( \frac{\partial \psi_t}{\partial t} = \mathbf{u} \) we have

\[
\mathbf{u}(\psi_t(\mathbf{x}(s), t)) \cdot \frac{\partial}{\partial t} \frac{\partial}{\partial s} \psi_t(\mathbf{x}(s)) = \mathbf{u}(\psi_t(\mathbf{x}(s), t)) \cdot \frac{\partial}{\partial s} \mathbf{u}(\psi_t(\mathbf{x}(s), t)) = \frac{1}{2} \frac{\partial}{\partial s} (\mathbf{u} \cdot \mathbf{u})(\psi_t(\mathbf{x}(s), t))
\]
\[ \Rightarrow \frac{d}{dt} \int \mathbf{u} \cdot d\mathbf{l} = \int_0^1 \frac{D\mathbf{u}}{Dt} (\psi_t(x(s), t)) \cdot \frac{d}{ds} \psi_t(x(s), t) \, ds \]

\[ = \int_C \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{l} \]

Employing the Euler equations:

\[ \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p - \nabla V \]

we obtain

\[ \frac{d}{dt} \int \mathbf{u} \cdot d\mathbf{l} = \int_C -\left( \frac{1}{\rho} \nabla p + \nabla V \right) \cdot d\mathbf{l} \]

\[ = - \int_C \nabla \left( \int \frac{d\mathbf{r}}{\rho(r)} + V \right) \cdot d\mathbf{l} \]

(provided all variables

are single-valued) \Rightarrow 0
Using Stokes' theorem

\[\Gamma_C = \oint_C \mathbf{u} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{u}) \cdot d\mathbf{S} = \iiint_S \mathbf{\omega} \cdot d\mathbf{S}\]

Definition: A vortex sheet (or line) is a surface \( S \) (or a curve \( L \)) that is tangent everywhere to the vorticity vector \( \mathbf{\omega} \).

Vortex sheets (or lines) are materially advected:

Consider \( \mathcal{S} \subseteq \mathbb{R}^3 \), then by Kelvin's circulation theorem:

\[\Gamma = \iint_S \mathbf{\omega} \cdot d\mathbf{S} = 0\]

since \( \mathbf{\omega} \cdot d\mathbf{S} = 0 \) at \( t = 0 \)

\[\Rightarrow \iint_{\mathcal{S}_t} \mathbf{\omega} \cdot d\mathbf{S} = 0\] at \( t > 0 \)

\[\Rightarrow \mathbf{\omega} \cdot d\mathbf{S} = 0\] at \( t > 0 \)

\[\Rightarrow \mathcal{S}_t \text{ is a vortex sheet.}\]
If \( \mathbf{\Omega}(\mathbf{x}) \neq 0 \), locally, a vortex line is the intersection of two vortex sheets.

**Definition:** Vortex tube

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**Helmholtz's Theorem** for a barotropic fluid

(a) If \( c_1 \) and \( c_2 \) are any two curves encircling the vortex tube, then

\[
\oint_{c_1} \mathbf{u} \cdot d\mathbf{l} = \oint_{c_2} \mathbf{u} \cdot d\mathbf{l} = K \quad \text{(Vortex tube strength)}
\]

(b) \( \frac{dK}{dt} = 0 \) \( \Rightarrow \) the tube is materially advected

**Proof:**

Note \( S \) is a vortex sheet

\[
\Sigma: S_{11} \cup S_{12}
\]

Gauss Theorem: \( 0 = \int \nabla \cdot \mathbf{u} \, d\mathbf{x} = \int \mathbf{u} \cdot d\mathbf{A} = \int_{S_{11}} \mathbf{u} \cdot d\mathbf{A} + \int_{S_{12}} \mathbf{u} \cdot d\mathbf{A} \quad \text{Volume of tube} \]
\[ \Rightarrow \int_{S_1} \vec{w} \cdot d\vec{A} = -\int_{S_2} \vec{w} \cdot d\vec{A} \]

\[ \int_{C_1} \vec{u} \cdot d\vec{l} = +\int_{C_2} \vec{u} \cdot d\vec{l} \]

(curls are oriented and so are d\vec{A}s.)

(b) follow directly from Kelvin's circulation theorem.

Remarks.

* Stretching of a vortex tube decreases its cross-sectional area, and magnitude of \( \vec{w} \) increases (but \( \vec{w} \) cannot be generated by stretching).

* Smoke rings are manifestations of materially advected vortex tubes.

* Tornadoes

Shear flow
Generation of lift on an airfoil

at $t = 0$

$\Gamma = \frac{\delta l}{c_0}$

$0 < t < 1$

viscous boundary layer

Stationary Vortex

$t \gg 1$

$\Gamma = \frac{\delta l}{c_t}$

vortex shedding

Kelvin: $\Gamma = \frac{\delta l}{c_t}$

If $C_0$ is far from any wing, then $C_t$ can be viewed as a materially advected $C_0$. 