2) Low Reynolds number flow \( \text{Re} = \frac{UL}{\nu} \)

\( \text{Re} \ll 1 : \) "very small length scale, creeping flow, very viscous"

\[
\begin{cases}
\text{Re} \left( \frac{\partial \vec{u}}{\partial \xi} + \vec{u} \cdot \nabla \vec{u} + \nabla p \right) = \nabla \cdot \vec{u} \\
\nabla \cdot \vec{u} = 0
\end{cases}
\]

Remark: pressure cannot be eliminated on physical grounds since it transmits forces through the fluid.

If we had scaled \( p^* = \frac{p}{\nu L} \) rather than \( p^* = \frac{P}{\rho U^2} \), we would obtain:

\[
\begin{cases}
\text{Re} \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) + \nabla \rho = \nabla \cdot \vec{u} \\
\nabla \cdot \vec{u} = 0
\end{cases}
\]

\[
\Rightarrow -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u} = 0 \quad \text{Stokes flow}
\]

(Problem: see Children's Chapter 7)
Applications

The NS equation is in general not solvable.
In some special situations, one can make progress:

Suppose the flow is uni-directional and depends only on those coordinates which are transverse to that direction.

\[ \vec{u} \cdot \nabla \vec{u} = u_x^2 \quad \vec{u} = 0 \]

Then the NS equation become linear!

1) Couette flow

\[ \begin{align*}
\begin{array}{c}
\text{\vec{u} = 0} \\
\text{\vec{u} = (U, 0, 0)} \\
U = U(y)
\end{array}
\end{align*} \]

Assume flow is steady ("wait long enough" and assume a steady flow exist - no instabilities)

\[ \frac{-\nabla P}{P} + \nu \frac{\partial^2 u}{\partial y^2} = 0 \]

Suppose the flow is driven entirely by the motion of the boundary, and the pressures are equal for up- and downstream.
\[
\frac{d^2 U}{dy^2} = 0 \implies U = Ay + B
\]

Two boundary conditions:

\[
U = 0 \text{ at } x = 0 \implies B = 0
\]

\[
U = U_L \text{ at } x = L \implies A = \frac{U_L}{L}
\]

\[
U = \frac{U_L y}{L} \quad \text{Couette Flow}
\]

2) **Poiseuille (pipe) flow**

\[
\begin{align*}
\tau & \quad \rightarrow \vec{U} = (0, 0, u_\theta) \\
\rho & \quad \text{at } z = 0 \\
\rho & \quad \text{at } z = L \\
\end{align*}
\]

\[
P_L > P_0
\]

Axisymmetric steady flow: \( \frac{\partial u}{\partial r} = 0 \), \( \frac{\partial \theta}{\partial r} = 0 \)

Again: flow is in a channel, but depends on traverse coordinate \( r \) only \( \implies \vec{u} \cdot \nabla \vec{u} = 0 \)

\( r \)-component:

\[
0 = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left\{ \frac{\partial^2 u_r}{\partial r^2} \right\} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left( \frac{\partial^2 u_r}{\partial r^2} \right)
\]

\( z \)-component:

\[
0 = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left\{ \frac{\partial^2 u_z}{\partial z^2} \right\} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2 u_z}{\partial z^2} \right)
\]
\( \tau \)-component:

\[ p = p(z) \]

\( t \)-component:

\[ \frac{\partial p}{\partial t} = \text{const.} \quad \text{and} \quad \frac{1}{\tau} \frac{\partial}{\partial \tau} \left( \tau \frac{\partial U}{\partial \tau} \right) = \text{const.} \]

\[ \frac{\partial p}{\partial t} = \text{const.} \quad \text{and} \quad \frac{1}{\tau} \frac{\partial}{\partial \tau} \left( \tau \frac{\partial U}{\partial \tau} \right) = \text{const.} \]

\[ p(z) = p_0 + Cz \quad \text{with} \quad p(0) = p_0 \quad \text{and} \quad p(L) = p_L \]

\[ = p_0 + \frac{p_L - p_0}{L} z \quad (G < 0) \]

\[ \frac{U}{\mu} = \frac{1}{\tau} \frac{\partial}{\partial \tau} \left( \tau \frac{\partial U}{\partial \tau} \right) \]

\[ \tau \frac{\partial U}{\partial \tau} = \frac{1}{2} \frac{U}{\mu} r^2 + C \]

To ensure \( \frac{\partial U}{\partial \tau} \) finite for \( \tau \to 0 \), we require \( C = 0 \).

\[ \frac{\partial U}{\partial \tau} = \frac{1}{2} \frac{U}{\mu} r \quad \Rightarrow \quad U = C \theta + \frac{1}{2} \frac{U}{\mu} r^2 \]

No-slip boundary condition at pipe wall:

\[ U(r = a) = 0 \]

\[ C = -\frac{1}{4} \frac{U}{\mu} a^2 \]

\[ U = \frac{5}{4\mu} (r^2 - a^2) \geq 0 \]

Parabolic flow profile.
The force on the pipe wall due to the shear is given by

$$\sigma_{\eta} \cdot 2\pi a \cdot L = \mu \left. \frac{du}{d\eta} \right|_{\eta=0} \cdot 2\pi a L$$

$$= \frac{1}{8} Ga \cdot 2\pi aL = \pi Ga L$$

$$= \pi a^2 (p_L - p_0)$$

So balances the applied pressure force.

The velocity is maximal at centre of the pipe with $\eta=0$

$$U_{\text{max}} = \frac{Ga^2}{4\mu}$$

The flow rate through the pipe is

$$Q = \int_U dA = \frac{G}{4\mu} \int_0^a 2\pi r dr (r^2 - a^2)$$

$$= -\pi \frac{G}{8\mu} a^4 > 0$$

Homage: Don’t eat at McDonald’s!

Remark: For flow with Reynolds number above a critical Reynolds number $Re^* = \left( \frac{2aU_{\text{max}}}{\mu} \right)^*$, the flow becomes unstable. Transition from laminar flow to turbulence.