The velocity potential

Consider **irrotational flow**, i.e. \( \nabla \times \mathbf{u} = 0 \) (see later) "Zero vorticity"

For ideal flows this holds for all times if the flow is irrotational at time 0.

**Definition**: An irrotational ideal fluid is called potential flow.

For irrotational flow in a simply connected domain \( D \), there is a scalar function \( \Phi(x,t) \) on \( D \) such that for each \( t \)

\[
\mathbf{u} = \nabla \Phi \quad \Phi \text{ is called velocity potential}
\]

* easier to work with than \( \mathbf{u} \) (scalar vs vector)

If additionally the flow is incompressible with \( \nabla \cdot \mathbf{u} = 0 \)

\[
\Delta \Phi = 0 \quad (\text{Laplace equation})
\]

The flow is called **potential flow**

Note flow can be non-steady!

Boundary condition at impermeable wall: \( \frac{\partial \Phi}{\partial n} = 0 \).

Hence instantaneous flow pattern depends only on instantaneous boundary condition (not on history of flow). Flow response is immediate mediated by pressure (so valid if sound waves can propagate through domain in typical time of few days).
Bernoulli's Theorem  (Daniel Bernoulli  1738)

Assume an ideal fluid:

Each equation:

$$p \left( \frac{du}{dt} + u \cdot \nabla u \right) = -\nabla p - \rho \nabla V$$

Use $u \cdot \nabla u = \nabla \left( \frac{1}{2} u^2 \right) - u \times (\nabla \times u)$

For simplicity consider steady flow with $\frac{du}{dt} = 0$ and irrotational flow with $\nabla \times u = 0$. With constant density $\rho$:

$$\nabla \left( \frac{1}{2} u^2 \right) = -\frac{\nabla p}{\rho} - \nabla V$$

Integrate:

$$\frac{p}{\rho} + \frac{1}{2} u^2 + V = \text{const.}$$

(Growth: $V = g z$)  
(High-speed implies low pressure)

Now relax the irrotational constraint:

$$\nabla \left( \frac{1}{2} u^2 \right) - u \times (\nabla \times u) + \frac{\nabla p}{\rho} + \nabla V = 0$$

$$\implies u \cdot \nabla \left[ \frac{1}{2} u^2 + \frac{p}{\rho} + V \right] = 0$$

i.e. $\frac{p}{\rho} + \frac{1}{2} u^2 + V$ is constant along stream lines

(but can vary from one stream line to another)
**Example:** Pitot tube

Pressure measurement to measure fluid flow velocity on aircrafts.

\[ \frac{\partial u}{\partial z} = 0 \Rightarrow u = 0 \]

(Consider irrotational flow)

So at station 2 where we measure pressure \( p_2 \):

\[ \frac{p_1}{p} + 0 = \frac{p_2}{p} + \frac{1}{2} u^2 \quad \Rightarrow \quad u = \sqrt{\frac{2}{\rho} (p_1 - p_2)} \]

Simplifications:

* ignores viscosity and existence of "boundary layer"

but pressure must be continuous across boundary layers, otherwise there would be an unphysical infinite force.

* neglects "circulation" around wing
Since we talk about limitations of the assumption of an ideal fluid:

Consider a channel filled with fluid

\[ \begin{align*}
\text{fluid is pushed from left to right.}
\end{align*} \]

10. Euler's equation: \( \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} \)

incompressible: \( \frac{\partial u}{\partial x} = 0 \)

\[ \Rightarrow \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \Rightarrow \frac{\partial^2 p}{\partial x^2} = 0 \]

\[ \Rightarrow p(x) = p_1 - \frac{p_1 - p_2}{L} x \]

\[ \Rightarrow u = \frac{p_1 - p_2}{\rho L} x + u(0) \]

So a constant pressure difference accelerates the fluid to infinite velocity. (Recall the "slippery" nature. No tangential force at the walls can control the flow \( \Rightarrow \) we need "viscosity".)