Exercise 1 Consider a very large lake with volume $V \gg 1$. The lake is filled with one species of fish. For a large volume $V$ (why?) we can employ a continuum description of the fish. Derive an equation for the fish density $n(x, t)$ using the same idea as for the derivation of the continuity equation for a fluid; the number of fish is, of course, not conserved. Why do we need to consider a large lake?

Exercise 2 Show that for a square $n \times n$ matrix $A(t)$ which depends on a parameter $t$ we have

$$
\frac{d}{dt} \det A = \begin{vmatrix}
\frac{d}{dt} A_{11} & \frac{d}{dt} A_{12} & \cdots & \frac{d}{dt} A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{vmatrix}
+ \begin{vmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
\frac{d}{dt} A_{21} & \frac{d}{dt} A_{22} & \cdots & \frac{d}{dt} A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{vmatrix} + \cdots
$$

You may use the multi-linearity of the determinant and work with the definition of the derivative as $dA(t)/dt = \lim_{\epsilon \to 0} (A(t) - A(t - \epsilon))/\epsilon$.

Exercise 3 Gerstner waves were first discovered in 1802 by Frantisek Joseph Gerstner and are an example where the analytical expression can be readily given in Lagrangian framework whereas there is no explicit solution in the Eulerian framework. The Gerstner solutions are given by

$$
X(a, b, t) = a + \frac{1}{k} e^{kb} \sin k(a + ct)
$$
$$
Y(a, b, t) = b - \frac{1}{k} e^{kb} \cos k(a + ct),
$$

where $k, c$ are constants.
(i) Show that the Lagrangian coordinates \((a, b)\) do not correspond to an Eulerian location \((x, y)\) at any time. What is the geometric meaning of \((a, b)\) here?

(ii) Show that the determinant of the Jacobian \(\det J\) is an invariant.

**Exercise 4** Suppose \(S_t = \Phi^t_0 S_0\) is a region of fluid particles where \(\Phi^t_0\) is the Lagrange-to-Euler map. Consider a scalar function \(f(x, t)\) and its volume integral \(F = \int_{S_t} f(x, t) dV\) with \(dV = dx_1 dx_2 \cdots dx_n\). By calculating \(dF/dt\) derive the convection theorem (or also known as Reynold’s Transport Theorem)

\[
\frac{dF}{dt} = \int_{S_t} \left( \frac{Df}{Dt} + f \text{div}(u) \right) dV.
\]

Now consider a fixed finite region \(R\) with boundary \(\partial R\) and calculate \(\frac{d}{dt} \int_R f dV\). Interpret the difference between the two cases.

**Exercise 5** Consider the velocity field \((u, v) = (y, -x + \epsilon \cos \omega t)\). Find explicit expressions for the instantaneous streamlines, the Lagrangian particle path and the streak lines.