Example Sheet 1 Solutions

1. Simplest: write \( \mathbf{u} = \nabla \times (0, 0, \psi(r, \theta)) \)
and use the form for curl in cylindrical polars
\[
\nabla \times \mathbf{A} = \frac{1}{R} \begin{vmatrix}
\hat{e}_r & \hat{e}_\phi & \hat{e}_z \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\
A_r & R A_\phi & A_z
\end{vmatrix}
\]

with \( R_{\text{cyl. pols}} \rightarrow R_{\text{plane pols}} \), \( \phi_{\text{cyl. pols}} \rightarrow \Theta_{\text{plane pols}} \)

\[ u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}, \quad \text{directly.} \]

Alternative (from first principles)

\[ u_r = u_x \cos \Theta + u_y \sin \Theta \]
\[ u_\theta = -u_x \sin \Theta + u_y \cos \Theta \]

Using the chain rule

\[ u_x = \frac{\partial y}{\partial y} = \frac{\partial y}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial y}{\partial \Theta} \frac{\partial \Theta}{\partial x} \]
\[ u_y = -\frac{\partial x}{\partial y} = -\frac{\partial x}{\partial r} \frac{\partial r}{\partial y} - \frac{\partial x}{\partial \Theta} \frac{\partial \Theta}{\partial y} \]
\[ r^2 = x^2 + y^2 \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos \Theta, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin \Theta \]
\[ \Theta = \tan^{-1} \frac{y}{x} \Rightarrow \frac{\partial \Theta}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial \Theta}{\partial y} = \frac{x}{x^2 + y^2} \]

\[ u_r = \left( \frac{\partial y}{\partial r} \sin \Theta + \frac{\partial y}{\partial \Theta} \cos \Theta \right) \frac{\cos \Theta}{r} - \left( \frac{\partial x}{\partial r} \cos \Theta - \frac{\partial x}{\partial \Theta} \sin \Theta \right) \frac{\sin \Theta}{r} \]
\[ u_\theta = -\left( \frac{\partial x}{\partial r} \sin \Theta + \frac{\partial x}{\partial \Theta} \cos \Theta \right) \frac{\cos \Theta}{r} + \left( \frac{\partial y}{\partial r} \cos \Theta - \frac{\partial y}{\partial \Theta} \sin \Theta \right) \frac{\sin \Theta}{r} \]

which gives \( u_r = \frac{1}{r} \frac{\partial \psi}{\partial \Theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r} \).
2. The Stokes stream function is defined by

\[ u_r = -\frac{1}{R} \frac{\partial \Psi}{\partial \eta} , \quad u_\eta = \frac{1}{R} \frac{\partial \Psi}{\partial \xi} \tag{\*} \]

Integrating the second of these with respect to \( \xi \),

\[ \Psi(R, \xi) = \int_0^R \eta u_\eta(\eta, \xi) d\eta + F(\xi) \]

where \( F(\xi) \) is (as yet) arbitrary.

Differentiate with respect to \( \xi \):

\[ \frac{\partial \Psi}{\partial \xi} = \int_0^R \eta \frac{\partial u_\eta(\eta, \xi)}{\partial \xi} d\eta + F'(\xi) \]

But \( \nabla \cdot u = 0 \) \( \Rightarrow \) \[ \int_0^R \eta \frac{\partial u_\eta(\eta, \xi)}{\partial \xi} d\eta = 0 \]

(remembering \( \eta \) is a dummy variable for \( R \))

\[ \frac{\partial \Psi}{\partial \xi} = -\int_0^R \frac{\partial}{\partial \eta} \left[ u_R(\eta, \xi) \eta \right] d\eta + F'(\xi) \]

\[ = -R u_R(R, \xi) + a u_R(a, \xi) + F'(\xi) \]

But from (\*), \[ \frac{\partial \Psi}{\partial \xi} = -R u_R(R, \xi) \]

so \( F(\xi) = -\int_0^a a u_R(a, \xi) d\xi \)

\[ \Rightarrow \Psi(R, \xi) = \int_0^R \eta u_\eta(\eta, \xi) d\eta - \int_0^a a u_R(a, \xi) d\xi \]

\[ \xi = b \tag{2} \]

\[ \xi = b \]

\[ \Psi \text{ is the volume flux across the area directed upwards (1)} \]

\[ + \text{the volume flux across the cylindrical face directed inwards (2)} \]

By continuity/incompressibility, this is the flux through any curve joining \((a, b)\) to \((R, \xi)\), when rotated about the axis to form a surface of revolution.
3. Simplest is to use the definition in terms of vector potential \( \mathbf{u} = \nabla \times (0, \frac{\Phi(R, \theta)}{R}, 0) \) in cylindrical polar \((R, \phi, z)\).

The vector \((0, \frac{\Phi(R, \theta)}{R}, 0)\) can be expressed as \((0, 0, \frac{\Phi(r \theta)}{rsin\theta})\) in spherical polar \((r, \theta, \phi)\).

Use of the formula for curl in spherical polar (see back of the notes) then gives

\[
\mathbf{u} = \left\{ \begin{array}{c}
\frac{1}{rsin\theta} \frac{\partial}{\partial \theta} \left( \frac{\Phi}{rsin\theta} \right), \\
-\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\Phi}{rsin\theta} \right), \\
0
\end{array} \right\}
\]

\[
= \left( \frac{\Phi}{r^2 sin\theta}, -\frac{1}{rsin\theta} \frac{\partial \Phi}{\partial r}, 0 \right).
\]

(One could also use the chain rule, though I don't recommend it)

4. \( u = UPx^{-1/3} \text{sech}^2(ax^{-2/3}) \)

[Diagram of velocity profile of \( u \)]

\( u = \frac{\partial \psi}{\partial y} \Rightarrow \psi = UPx^{-1/3} \text{tanh}(ax^{-2/3}) \)

[Equation for \( \psi \)]

+ an arbitrary function of \( x \).

By choosing \( \psi = 0 \) to be the streamline on the line of symmetry \( y = 0 \), we can compel the arbitrary function of \( x \) to be zero, and then

\[
\psi = \frac{U}{\alpha} x^{1/3} \text{tanh}(ax^{-2/3}).
\]

Then the \( y \)-component \( v = -\frac{\partial \psi}{\partial x} \)

[Equation for \( v \)]

\[
= -U \frac{x^{-2/3}}{\alpha} \text{tanh}(ax^{-2/3}) + \frac{2}{3} y U \frac{4}{\alpha} \frac{x^{-4/3}}{\alpha} \text{sech}^2(ax^{-2/3}).
\]
5. \[ \psi = r U \left( \Theta \cos \Theta + \frac{\pi}{2} \Theta \sin \Theta - \sin \Theta \right) \]

\[ u_r = \frac{1}{r} \frac{\partial \psi}{\partial r} = U \left( \cos \Theta - \Theta \sin \Theta + \frac{\pi}{2} \left( \sin \Theta + \Theta \cos \Theta \right) - \cos \Theta \right) \]

\[ u_\theta = -\frac{\partial \psi}{\partial \theta} = -U \left( \Theta \cos \Theta + \frac{\pi}{2} \Theta \sin \Theta - \sin \Theta \right), \quad \text{at } \theta = 0 \text{ or } \frac{\pi}{2} \]

Since \[ \frac{\pi^2}{4} \approx 2.5 > 1 \], the velocity at \( \theta = \frac{\pi}{2} \) is negative and constant, i.e., it is \( U \left( \frac{\pi^2}{4} - 1 \right) \)
in the positive x-direction.

So this might model a paint-scraper being moved in the x-direction at this speed, at some instantaneous moment in time.

\[ U \left( \frac{\pi^2}{4} - 1 \right) \]

\[ u \text{ tangential} = 0 \text{ on } \theta = 0 \]

and \( \theta = \frac{\pi}{2} \), so no-slip BCs apply.

In the instantaneus origin.

\[ \nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \]

\[ = U \left( \Theta \cos \Theta + \frac{\pi}{2} \Theta \sin \Theta - \sin \Theta \right) + U \cdot r \cdot \frac{\mid \frac{\pi}{2} \cos \Theta - \sin \Theta \mid}{r^2} \]

\[ = 2U \frac{\mid \frac{\pi}{2} \cos \Theta - \sin \Theta \mid}{r} \]

\[ \nabla^4 \psi = \nabla^2 (\nabla^2 \psi) \text{ has contributions proportional to } \nabla^2 \left( r \frac{\cos \Theta \ or \ \sin \Theta}{r} \right) \]

\[ \nabla^2 \left( \frac{\cos \Theta \ or \ \sin \Theta}{r} \right) = \left( \cos \Theta \ or \ \sin \Theta \right) \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \right) - \frac{1}{r^2} \left( \cos \Theta \ or \ \sin \Theta \right) \]

\[ = \left( \cos \Theta \ or \ \sin \Theta \right) \left\{ \frac{1}{r^3} - \frac{1}{r^3} \right\} = 0 \]. So \[ \nabla^4 \psi = 0 \].
6. The velocity potential for the second case was given in the lectures.

For the 2-D geometry,
\[ \Phi = \frac{S}{2\pi} \log r_1 + \frac{S}{2\pi} \log r_2 \]
\[ \text{ie } \Phi = \frac{S}{2\pi} \left[ \log \left( \frac{x^2+(y-h)\gamma^2}{x^2+(y+h)\gamma^2} \right) \right] \]
\[
\mathbf{u} = \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right) = \frac{S}{2\pi} \left[ \frac{x}{x^2+(y-h)^2} + \frac{x}{x^2+(y+h)^2}, \frac{y-h}{x^2+(y-h)^2} + \frac{y+h}{x^2+(y+h)^2} \right]
\]
on differentiating and using the chain rule.

Hence \( \mathbf{u} \) on the plane \( y = 0 \) is \( \frac{S}{2\pi} \left[ \frac{2x}{x^2+h^2}, 0 \right] \).

Thus the y-component is zero, verifying that the use of the image satisfies the zero normal flow BC, and the x-component is zero at \( x = 0 \) and \( x \to \pm \infty \).

The maximum speed is developed where \( \frac{d}{dx} \left\{ \frac{x}{x^2+h^2} \right\} = 0 \),
\ie\ where \( \frac{1}{x^2+h^2} - \frac{2x^2}{(x^2+h^2)^2} = 0 \), \ie\ \( x^2 = h^2 \), \ie\ \( x = \pm h \).

For the 3-D case (which is axisymmetric)
\[ \Phi = -\frac{S}{4\pi} \left[ \frac{1}{\sqrt{R^2+(z-h)^2}} + \frac{1}{\sqrt{R^2+(z+h)^2}} \right] \]
(\text{notes p16, writing } x^2+y^2 = R^2.)

\[ \mathbf{u} = \left( \frac{\partial \Phi}{\partial R}, 0, \frac{\partial \Phi}{\partial z} \right) \text{ in cylindrical polar} \]
\[ = \frac{S}{4\pi} \left[ \frac{R}{\left( R^2+(z-h)^2 \right)^{3/2}}, \frac{R}{\left( R^2+(z+h)^2 \right)^{3/2}}, 0, \frac{z-h}{\left( R^2+(z-h)^2 \right)^{3/2}}, \frac{z+h}{\left( R^2+(z+h)^2 \right)^{3/2}} \right] \]
\[ = \frac{S}{4\pi} \left[ \frac{2R}{\left( R^2+h^2 \right)^{3/2}}, 0, 0 \right] \text{ on the plane } z = 0. \]

Again, the \( z \)-component is zero as it should be, and the \( R \)-component is zero at \( R = 0 \), \( R \to \infty \). The maximum speed is where \( \frac{d}{dR} \left( \frac{R}{\left( R^2+h^2 \right)^{3/2}} \right) = 0 \), \ie\ when \( \frac{R^2+h^2-3R^2}{(R^2+h^2)^{5/2}} = 0 \), \ie\ \( R = \frac{h}{\sqrt{2}} \).
The problem is axisymmetric, suggesting we use cylindrical polar coordinates. In that coordinate system, the SSF for the uniform stream is $\Psi = \frac{1}{2} UR^2$.

A source at the origin has $\text{SSF } \Psi = -\frac{S \cos \theta}{4 \pi}$, in spherical polar coordinates. Thus at our test point $(r_1, \theta, \phi)/ (R, \phi, z)$ (in mixed, or "schizoid" coordinates), the sources $\pm S$ give contributions to $\Psi$ of $-\frac{S \cos \theta_1}{4 \pi}$ and $+\frac{S \cos \theta_2}{4 \pi}$ respectively.

We had better de-schizophrenise our coordinates, so let's go for cylindrical polar.

Then $\cos \theta_1 = \frac{z+a}{r_1} = \frac{z+a}{\sqrt{r_1^2 + (z+a)^2}}$, $\cos \theta_2 = \frac{z-a}{r_2} = \frac{z-a}{\sqrt{r_2^2 + (z-a)^2}}$.

\[ \Psi_{\text{total}} = \frac{1}{2} UR^2 - \frac{S}{4 \pi} \left( \frac{z+a}{\sqrt{r_1^2 + (z+a)^2}} - \frac{z-a}{\sqrt{r_2^2 + (z-a)^2}} \right) \]

There will be two stagnation points at $A$ and $B$, with $\Psi$ defined as above, these lie on the $\Psi = 0$ streamline. At the stagnation points, the $\Psi = 0$ streamline bifurcates, and is given by

\[ R = \left( \frac{S}{2 \pi U} \right)^{1/2} \left\{ \frac{z+a}{[R^2 + (z+a)^2]^{1/2}} - \frac{z-a}{[R^2 + (z-a)^2]^{1/2}} \right\}^{1/2} \]

This curve gives the shape of the so-called Rankine Ovoid, which is a streamline which can be replaced by a solid body.
At the stagnation points, the total velocity vanishes. Suppose the leftmost one $A$ is at $z = -\lambda$, the rightmost one $B$ is at $z = +\lambda$. Then the velocities at $A$ and $B$ due to the various components are directed along the $z$-axis as follows:

\[ \frac{S}{4\pi(a^2)} \quad \text{from} \quad +S \quad \rightarrow \quad U \]

\[ \frac{S}{4\pi(a^2 + \lambda)^2} \quad \text{from} \quad -S \quad \rightarrow \quad U \]

\[ \frac{S}{4\pi(a + \lambda)^2} \quad \text{from} \quad +S \quad \rightarrow \quad \frac{S}{4\pi(a + \lambda)^2} \quad \text{from} \quad -S \]

For a stagnation point, both of these give $\frac{4\pi U}{S} = \frac{1}{(\lambda - a)^2} - \frac{1}{(\lambda + a)^2}$, or \( L^2 - 1 = \frac{S}{4\pi a^2 U} \),

where \( L = \frac{\lambda}{a} \) is dimensionless and \( \frac{S}{4\pi a^2 U} \) is a dimensionless parameter measuring the relative strength of the source to the stream. This has one root with \( L > 1 \), as can be seen graphically on the left.

Clearly by symmetry the radius is greatest when \( z = 0 \), and then \( R_{\text{max}} = \left( \frac{S}{4\pi U} \right) \left[ \frac{2a}{(R_{\text{max}} + a)^2} \right] \), ie $\frac{4\pi U}{S} \sqrt{R_{\text{max}} + a} \frac{R_{\text{max}}^2}{\pi a} = 1$.

Returning to spherical points, with $R = r\sin\theta$, $z = r\cos\theta$, the shape of the ovoid \( \bigcirc \) converts to \( \bigotimes \).

\[ r^2 \sin^2 \theta = \frac{S}{4\pi U} \left\{ \frac{\cos\theta + a}{(r^2 + 2ar\cos\theta + a^2)^{3/2}} - \frac{r\cos\theta - a}{(r^2 - 2ar\cos\theta + a^2)^{3/2}} \right\} \]

\[ = \frac{S}{4\pi U} \left\{ (\cos\theta + a)(1 + r\cos\theta + a^2)^{1/2} - (\cos\theta - a)(1 - r\cos\theta + a^2)^{1/2} \right\} \]

For small $a$, neglect terms of order $a^3/2$ and binomially expand

\[ r^2 \sin^2 \theta = \frac{S}{4\pi U} \left[ (\cos\theta + a(1 - \cos^2\theta) + \ldots) - (\cos\theta + a(1 - \cos^2\theta) + \ldots) \right] - \frac{1}{\pi} \]

\[ \Rightarrow r^2 \sin^2 \theta = \frac{S}{4\pi U} \frac{2a}{\sin^2 \theta} \text{... as } a \to 0 \]

\[ r = \left( \frac{Sa}{4\pi U} \right)^{1/2} \text{ a sphere, for } Sa \text{ finite} \]
8. The flow is spherically symmetric and incompressible, so that \( \nabla \cdot u = 0 \). In spherical polar \((r, \theta, \phi)\) with \( \frac{\partial}{\partial r} = \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} \equiv 0 \), this gives
\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u(r,t)) = 0,
\]
where \( u = (u(r,t), 0, 0) \).

\( \Rightarrow r^2 u(r,t) = f(t) \).

The RHS of this can depend on the parameters \( E \) and \( p \) as well as time, but not on anything else.

So try
\[
r^2 u(r,t) = KE^\alpha p^\beta t^\gamma,
\]
where \( K \) is dimensionless, and use dimensional analysis to work out \( \alpha, \beta, \gamma \).

- \([r^2] = L^2\);
- \([u] = LT^{-1}\);
- \([E] = [M][V^2] = ML^2T^{-2}\);
- \([p] = [M]_\text{Vol} = ML^{-3}\);
- \([t] = T\).

Hence \( L^3T^{-1} = (ML^2T^{-2})^\alpha (ML^{-3})^\beta T^\gamma \). 

Length \( \Rightarrow 3 = 2\alpha - 3\beta \)

Mass \( \Rightarrow 0 = \alpha + \beta \)

Time \( \Rightarrow -1 = -2\alpha + \gamma \)

\( \Rightarrow \alpha = \frac{3}{5}, \beta = -\frac{3}{5}, \gamma = \frac{1}{5} \).

Hence \( r^2 u = K \left( \frac{E}{p} \right)^{3/5} t^{1/5} \), i.e. \( u = K \left( \frac{E}{p} \right)^{3/5} t^{1/5} r^2 \).

Apply this result at the boundary of the bubble \( r = R(t) \); hence, the velocity must be the rate of increase of the bubble radius, \( \frac{dR}{dt} \).

Hence \( \frac{dR}{dt} = K \left( \frac{E}{p} \right)^{3/5} t^{1/5} R^2(t) \), which can be integrated to give
\[
\frac{1}{3} R^3(t) = \frac{5}{6} K \left( \frac{E}{p} \right)^{3/5} t^{6/5} + \text{const.}
\]

At \( t = 0, R = 0 \), so const. = 0, and then
\[
R(t) = \left( \frac{5}{2} K \right) \left( \frac{E}{p} \right)^{1/5} t^{1/5}
\]
The image system shown has $V^2 \phi = 0$ in the physically relevant region ($\Phi > 0$) and has introduced no new singularities thereof.

Moreover, it is clear from the symmetry of the flows that there is now no normal flow across the boundary surfaces $x = 0, y = 0$, so the boundary condition is satisfied. By a uniqueness theorem, this image system gives the solution.

The line vortices have associated fluid speed $\pm \frac{K}{2\pi r}$ directed tangentially where $r$ is the distance from the vortex. Thus at the point $(a, b)$, the induced velocity arising from the image vortices is summarised in the following diagram

\[
\begin{array}{c}
\text{(from I2)} \\
\frac{\pi \cdot 2l}{2\pi r} \\
\frac{\pi \cdot 2b}{2\pi r} \\
\frac{\pi \cdot 2a}{2\pi r}
\end{array}
\]

Thus the $x$-component of the net velocity, which must equal $\frac{da}{dt}$, is

\[
\frac{K}{2\pi r} \cdot \sin \theta = \frac{da}{dt} = \frac{K}{4\pi} \left[ \frac{1}{b} - \frac{1}{\sqrt{a^2 + b^2}} \right].
\]

Similarly, the $y$-component gives

\[
\frac{db}{dt} = -\frac{K}{4\pi} \left[ \frac{1}{a} - \frac{1}{\sqrt{a^2 + b^2}} \cdot \frac{a}{\sqrt{a^2 + b^2}} \right],
\]

To find the path of the vortex, divide these to give

\[
\frac{da}{db} = \frac{a^2}{b^2} \cdot \frac{-b^2}{a^3},
\]

so

\[
\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{a_0^2}
\]

If $a = a_0$ as $b \to \infty$, the constant is $-\frac{1}{a_0^2}$.
10. Unsteady irrotational flow has \( u = \nabla \Phi \) (because \( \nabla \times u = 0 \))

and \( \frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla p}{\rho} - \nabla V \)

Use the identity \( u \cdot \nabla u = \nabla \left( \frac{1}{2} u^2 \right) - u \times \omega \)

to give \( \nabla \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} u^2 + V \right) = -\frac{\nabla p}{\rho} \).

Now \( \int \frac{dp}{\rho} \) is only well-defined if \( \rho = \rho(p) \),
and then \( \int \frac{dp}{\rho(p)} = \frac{1}{p} \nabla p \), by the chain rule for each component.

Then "unreducing" the resultant equation gives

\[
\frac{\partial \Phi}{\partial t} + \frac{1}{2} u^2 + V + \int \frac{dp}{\rho} = \text{function of } t.
\]

Since \( u = \nabla \Phi \) is unchanged by the addition of an arbitrary function of time, we can choose a gauge for \( \Phi \) which incorporates the function of time in the definition of \( \Phi \), giving

\[
\int \frac{dp}{\rho} + \frac{1}{2} u^2 + \frac{\partial \Phi}{\partial t} + V = \text{constant}.
\]

11. In cylindrical polars, if \( u = (0, u(r), 0) \),
then \( \omega = (0, 0, \frac{1}{r} \frac{\partial}{\partial r} (ru(r))) \) (using the expression for curl in cylindrical polars).

We have \( \frac{\partial}{\partial r} (ru) = ru \) (\( r < a \))

\( \Rightarrow ru = \frac{1}{2} ru^2 + A \) (\( r < a \))

\( \Rightarrow B \) (\( r > a \)).

For \( v \) to be finite at the origin, we must take \( A = 0 \).
If \( v \) is to be continuous at \( r = a \), \( B = \frac{1}{2} \rho a^2 \).

In \( r > a \), Bernoulli's equation is \( \rho + \frac{1}{2} \rho v^2 + \rho g z = \) const (including gravity via \( \Phi = g z \)). Take \( z = 0 \) to be the free surface height at \( r \to \infty \) with pressure \( p_0 \).

Thus \[ p = p_0 - \frac{1}{8} \rho \frac{a^4}{r^2} - \rho g \frac{a}{r} \quad \text{in} \quad r > a. \] (putting \( v = \frac{2a^2}{r^2} \))

For \( r < a \), the momentum equation can be written
\[ \rho \left(\nabla \left(\frac{1}{2} v^2\right) - \nabla \Phi\right) = -\nabla p - \rho \nabla \Phi \quad \text{(using Q12 part (i))} \]
Now \( \nabla \Phi = (0,0,0) \times (0,0,\nabla z) = \left(\frac{1}{2} x \partial_x^2, 0, 0\right) = \nabla \left(\frac{1}{2} x r^2\right) \)
in this particular case. Also \( v = \frac{1}{2} \rho a^2 r \).

So the above equation can be "ungraded" to give
\[ \rho \left(\frac{1}{8} \rho \frac{a^4}{r} - \frac{1}{2} \rho \frac{a^2}{r^2} \right) + p + \rho g z = \text{constant}. \]

i.e. \[ p = \text{constant} + \frac{1}{8} \rho \frac{a^4}{r} - \rho g \frac{a}{r}. \]

Now at \( r = a \), we must have \( p \) continuous (otherwise there would be infinite accelerations at the interface).

Thus the constant is \( p_0 - \frac{1}{4} \rho \frac{a^2}{a^2} \rho \).

For the free surface, put \( p = p_0 \) everywhere along it.

Then for \( r > a \),
\[ z = -\frac{1}{8} \rho \frac{a^4}{g r^2} \]
and for \( r < a \)
\[ z = \frac{1}{8} \rho \frac{a^2}{g} \left(r^2 - 2a^2\right) \]

This surface is continuous at \( r = a \) (reassuringly!); also its derivative is continuous. Its second derivative has an (unphysical) discontinuity. However, it is a much better model than a line vortex.