1. The angular momentum associated with the relative motion of the sphere about its centre is

\[ \rho \int \left( \mathbf{d\mathbf{r}} \times \mathbf{du} \right) dV \]

where \( \mathbf{d\mathbf{r}} \) is the vector going from the centre of the sphere to a point within it, and \( \mathbf{du} \) is the velocity of that point relative to the velocity of the centre.

Use

\[ du = \left( \frac{\partial u}{\partial x} \right)_c \, dx + \left( \frac{\partial u}{\partial y} \right)_c \, dy + \left( \frac{\partial u}{\partial z} \right)_c \, dz \]

\[ dv = \left( \frac{\partial v}{\partial x} \right)_c \, dx + \left( \frac{\partial v}{\partial y} \right)_c \, dy + \left( \frac{\partial v}{\partial z} \right)_c \, dz \]

\[ dw = \left( \frac{\partial w}{\partial x} \right)_c \, dx + \left( \frac{\partial w}{\partial y} \right)_c \, dy + \left( \frac{\partial w}{\partial z} \right)_c \, dz \]

to estimate the relative velocity by Taylor expansion about the centre \( C \); in the limit where the radius of the sphere tends to zero, the error associated with this becomes negligible. The partial derivatives \( \left( \frac{\partial u}{\partial x} \right)_c \) etc are evaluated at the centre of the sphere, and thus they are constants as far as the integration is concerned. We work out just the \( x \)-component of the relative angular momentum; the \( y \) and \( z \) components are analogous.
\{ \mathbf{dr} \times \mathbf{du} \}_x = dy \left[ \frac{\partial w}{\partial x} \mathbf{dx} + \frac{\partial w}{\partial y} \mathbf{dy} + \frac{\partial w}{\partial z} \mathbf{dz} \right] - dz \left[ \frac{\partial v}{\partial x} \mathbf{dx} + \frac{\partial v}{\partial y} \mathbf{dy} + \frac{\partial v}{\partial z} \mathbf{dz} \right].

When this expression is integrated over the sphere, there are two sorts of terms, ones like \( \frac{\partial w}{\partial x} \) \( \int \mathbf{dy} \mathbf{dx} \mathbf{dV} \) (there are four of these), and ones like \( \frac{\partial w}{\partial y} \) \( \int (dy)^2 \mathbf{dV} \) (there are two of these).

Because V is a sphere, the first type of integral gives zero because for each \( dx \) and \( dy \) there is an equal and opposite contribution from \( dx \) and \( -dy \) (say). Because the sphere is isotropic, it knows no difference between \( x, y, \) or \( z \), and the 3 integrals \( \int (dx)^2 \mathbf{dV}, \int (dy)^2 \mathbf{dV} \) and \( \int (dz)^2 \mathbf{dV} \) are all equal.

\[ 3 \mathbf{g} = \int \left[ \frac{(dx)^2}{r^2} + \frac{(dy)^2}{r^2} + \frac{(dz)^2}{r^2} \right] \mathbf{dV} = \int r^2 \mathbf{e}_r \mathbf{dV} \]

In fact, \( 3 \mathbf{g} = \int \left[ \frac{(dx)^2}{r^2} + \frac{(dy)^2}{r^2} + \frac{(dz)^2}{r^2} \right] \mathbf{dV} = \int r^2 \mathbf{e}_r \mathbf{dV} \)

on writing \( r^2 = (dx)^2 + (dy)^2 + (dz)^2 \) and using spherical polar with volume element \( dr \cdot d\mathbf{r} = r^2 \sin \theta \mathbf{d} \theta \mathbf{d} \phi \).

Thus \( 3 \mathbf{g} = \int r^4 \mathbf{d} r \int \sin \theta \mathbf{d} \theta \mathbf{d} \phi \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} = \frac{4}{5} \pi \mathbf{e}_r \)

Hence the \( x \)- component of the relative angular momentum is \( P \left[ \frac{\partial w}{\partial y} \right] \mathbf{g} = \frac{4}{15} \pi \mathbf{e}_x \)
evaluated at \( C \). Similarly for the other 2 components.

\( \frac{4}{3} \pi \mathbf{e}_r \) is the mass of the sphere. Thus the relative angular momentum is \( \frac{1}{5} \mathbf{e}_x \). But the moment of inertia \( I \) of the sphere considered as a rigid body would be \( \frac{2}{5} \mathbf{e}_x \). Thus the relative angular momentum is \( \frac{1}{2} \mathbf{e}_x \). A solid sphere rotating with \( \omega \) would have A.M. \( \frac{1}{2} I \), so \( \frac{1}{2} \omega \) is like a local rotation rate.
Mass conservation $\Rightarrow$ $pl_1 S_1 = pl_2 S_2$
as we follow a fixed element of fluid.
Thus $pl S$ is constant, equal to the mass in the element.
Kelvin's theorem $\Rightarrow k = \int u \cdot ds$ is constant following the fluid, i.e. constant circulation.
But this is $\int \nabla \times \mathbf{u} \cdot ds \equiv \mathbf{w} \cdot S$.
$\therefore \mathbf{w} \cdot S$ is constant (the circulation).
Dividing these, we have $\frac{\mathbf{w}}{pl}$ constant as we follow the element.
Thus increasing $l$ ("stretching" the tube) means $\mathbf{w}$ must increase by a corresponding amount, other things being equal.

This is similar to what happens in Q3. Note a first attempt to model a tornado might be to take one half (say $z > 0$) of the flow in Q3. The $\alpha$-part of the flow might be provided by a cumulonimbus convection cell. (Real tornadoes are far more complicated than this. If they were it, they would happen all over the place; the vortex stretching mechanism is widespread in the atmosphere. Something extra special is needed to cause a tornado.)
3. \( u = \left( -\frac{\alpha R}{2}, v(R,z,t), R, z \right) \) using cylindrical polars.

\[
\omega = \nabla \times u = \left( -\frac{\partial v}{\partial z}, 0, \frac{1}{R} \frac{\partial}{\partial R} (RV) \right)
\]

This has to be of the form \((0, 0, \omega)\), which shows \(\frac{\partial v}{\partial z} = 0 \Rightarrow v = v(R,t)\), and then

\[
\omega = \frac{1}{R} \frac{\partial}{\partial R} (RV(R,t)) = \omega(R,t).
\]

With viscosity negligible, the vorticity equation is

\[
\frac{\partial \omega}{\partial t} = \nabla \times (u \times \omega).
\]

Work out the RHS:

\[
u \times \omega = (v\omega, +\frac{\alpha R}{2} w, 0)
\]

\[
\nabla \times (u \times \omega) = (0, 0, \frac{1}{R} \frac{\partial}{\partial R} \left( R \cdot \frac{\alpha R}{2} w \right))
\]

(because \(v\omega\) is independent of \(z\)).

Hence the vorticity equation has 2 trivial components (our set-up is consistent), and the third component is

\[
\frac{\partial \omega}{\partial t} = \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{\alpha R}{2} w \right) = \frac{1}{R} \alpha R \frac{\partial \omega}{\partial R} + \omega(x) \tag{\star}
\]

Try \(\omega = w_0 e^{at} f(Re^{\frac{zt}{2}})\), where \(f\) is an arbitrary once-differentiable function.

Then \[
\frac{\partial \omega}{\partial t} = w_0 e^{at} f + w_0 R \alpha e^{\frac{zt}{2}} f' e^{at} = \omega + \alpha R w_0 e^{\frac{zt}{2}} \tag{\star'}
\]

\[
\frac{\partial \omega}{\partial R} = w_0 e^{at} f' e^{\frac{zt}{2}}
\]

So \(\star\) is clearly satisfied, and the proposed solution works.
Aside: to find the solution to the PDE directly, write it as
\[
\frac{\partial}{\partial t} (\frac{\partial}{\partial R} w) = 0, \text{ or } \left( \frac{\partial}{\partial t} - \frac{\alpha R}{2} \frac{\partial}{\partial R} \right) (e^{-\alpha t} w) = 0.
\]
This means that we \( e^{-\alpha t} \) is constant along each curve in the \((R,t)\) plane which satisfies \( \frac{dR}{dt} = -\frac{\alpha R}{2} \).

Because \( \frac{d}{dt} (g(R(t),t)) = \frac{dR}{dt} \frac{dg}{dR} + \frac{dg}{dt} \), where \( g \) is an arbitrary differentiable function, take \( g = w e^{-\alpha t} \).

These curves, called characteristics, have \( \log R = -\alpha t + \log C \), ie \( R e^{\alpha t/2} = C \), where each \( C \) value labels a different curve.

Thus \( w e^{-\alpha t} \) is a function of \( C = Re^{\alpha t/2} \), ie taking out an amplitude factor \( w_0 \),
\[
w(R,t) = w_0 e^{\alpha t} f(Re^{\alpha t/2}).
\]

This solution is an example of the method of characteristics for solving 1st-order PDEs — see MATH2965.

If at \( t=0 \) \( w = w_0 \) everywhere, so that \( f = 1 \), then at later times \( w = w_0 e^{\alpha t} \), so the vorticity intensifies exponentially in time due to vortex stretching.

Another example: if the vorticity at \( t=0 \) is localised in a patch of characteristic radius \( R_0 \), so that (for example) \( f(R) = e^{-(R^2/R_0^2)} \), then later
\[
w = w_0 e^{\alpha t} e^{-(R^2 e^{\alpha t}/R_0^2)}, \text{ so that the vortex is exponentially amplified in time but is confined to a core thickness } R_0 e^{-\alpha t/2} \text{ which decreases exponentially with } t.\]
4. To show \( u \cdot \nabla u = \nabla (\frac{1}{2} u^2) - u \times (\nabla \times u) \)

Start from \( \{ u \times (\nabla \times u) \}_i = \varepsilon_{ijk} u_j \frac{\partial u_m}{\partial x_j} \)

\[
= \varepsilon_{kij} \varepsilon_{kml} \frac{u_j}{\partial x_l} \frac{\partial u_m}{\partial x_i} \neq (\nabla \times u)_k \\
= (\delta_{il} \delta_{mj} - \delta_{im} \delta_{lj}) \frac{u_j}{\partial x_l} \frac{\partial u_m}{\partial x_i} \\
= \frac{u_j}{\partial x_l} - \frac{u_j}{\partial x_l} \frac{\partial u_l}{\partial x_j} = \frac{\partial}{\partial x_l} \left( \frac{1}{2} u_j u_l \right) - u_l \frac{\partial u_j}{\partial x_l} \\
= \left\{ \nabla \left( \frac{1}{2} u^2 \right) - (u \cdot \nabla u) \right\}_i \quad (i = 1, 2, 3) \\
\]

which proves the result.

To show \( \nabla \times (\nabla \times u) = \nabla (\nabla \cdot u) - \nabla^2 u \),

Start from \( \{ \nabla \times (\nabla \times u) \}_i = \varepsilon_{ijk} \frac{\partial }{\partial x_j} \left[ \varepsilon_{kml} \frac{\partial u_m}{\partial x_l} \right] \neq (\nabla \times u)_k \\
= \varepsilon_{kij} \varepsilon_{kml} \frac{\partial^2 u_m}{\partial x_j \partial x_l} \neq (\nabla \times u)_k \\
= (\delta_{il} \delta_{mj} - \delta_{im} \delta_{lj}) \frac{\partial^2 u_m}{\partial x_j \partial x_l} \\
= \frac{\partial}{\partial x_l} \left[ \frac{\partial u_j}{\partial x_l} \right] - \frac{\partial^2 u_l}{\partial x_j \partial x_l} \\
= \left\{ \nabla \left( \nabla \cdot u \right) - \nabla^2 u \right\}_i \\
\]

which proves the result.
Navier-Stokes for steady constant-density incompressible flow with conservative body forces can be written
\[ p \nabla u = -\nabla p - \rho \nabla V + \mu \nabla^2 u \]
where \( E = -\nabla V \) is the body force, \( V \) its potential.

Using the just-proved identities, this becomes (with \( \nabla \cdot u = 0 \) because of incompressibility)
\[ \rho \left[ \nabla \left( \frac{1}{2} u^2 \right) - u \times (\nabla \times u) \right] = -\nabla p - \rho \nabla V + \mu \left[ -\nabla \times (\nabla \times u) \right] \]
\( \nabla \times u = \omega \) is the vorticity, and because the density is uniform, \( \rho \) can be taken inside the gradient operator, to give
\[ \nabla \left( \frac{1}{2} \rho u^2 + p + \rho V \right) = \rho (u \times \omega) - \mu \nabla \times \omega. \]

Integrate this around any closed streamline within the flow; note \( \int \nabla \Phi \cdot dl = 0 \) for any scalar potential \( \Phi \) integrated around a closed curve, so the LHS gives zero. Also, on a streamline \( u \) and \( dl \) are parallel (this is just the defining property of a streamline), so \( (u \times \omega) \cdot dl \) can be written \( (u \times \omega) \cdot \lambda \) for some scalar field such that \( dl = \lambda u \), and this is zero.

Hence \( \mu \int \nabla \times \omega \cdot dl = 0. \)
5. Applying the relation given to the position vector $\mathbf{r}$ of a given fluid particle,

$$\left( \frac{\text{D}\mathbf{r}}{\text{D}t} \right)_{\text{IF}} = \left( \frac{\text{D}\mathbf{r}}{\text{D}t} \right)_{\text{RF}} + \mathbf{\Omega} \times \mathbf{r}$$

i.e. $\mathbf{u}_{\text{IF}} = \mathbf{u}_{\text{RF}} + \mathbf{\Omega} \times \mathbf{r}$.

Navier-Stokes in the inertial frame is

$$\rho \frac{\text{D}\mathbf{u}_{\text{IF}}}{\text{D}t} = -\nabla p + \rho \mathbf{F} + \mu \nabla^2 \mathbf{u}$$

Spatial derivatives are evaluated at fixed time and are the same whether the one is in the inertial or the rotating frame. Thus the equation becomes, in the rotating frame,

$$\rho \left\{ \frac{\text{D}}{\text{D}t} \right\}_{\text{RF}} \left( \mathbf{u}_{\text{RF}} + \mathbf{\Omega} \times \mathbf{r} \right) = -\nabla p + \rho \mathbf{F} + \nabla^2 \left( \mathbf{u}_{\text{RF}} + \mathbf{\Omega} \times \mathbf{r} \right)$$

$\mathbf{\Omega}$ is a constant vector, so $\nabla^2 (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{\Omega} \times \nabla^2 \mathbf{r} = 0$.

Hence $\rho \left\{ \frac{\text{D}\mathbf{u}_{\text{RF}}}{\text{D}t} + \mathbf{\Omega} \times \mathbf{u}_{\text{RF}} + \mathbf{\Omega} \times \frac{\text{D}\mathbf{r}}{\text{D}t} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \right\}$

$$= -\nabla p + \rho \mathbf{F} + \nabla^2 \mathbf{u}_{\text{RF}}$$

But $\frac{\text{D}\mathbf{r}}{\text{D}t} = \mathbf{u}_{\text{RF}}$, so we get

$$\rho \left( \frac{\text{D}\mathbf{u}_{\text{RF}}}{\text{D}t} + 2 \mathbf{\Omega} \times \mathbf{u}_{\text{RF}} \right) = -\rho \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) - \nabla p + \rho \mathbf{F} + \mu \nabla^2 \mathbf{u}_{\text{RF}}$$

$2 \rho \mathbf{\Omega} \times \mathbf{u}_{\text{RF}}$ (strictly, $-i\rho \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$) is called the (fictitious) Coriolis Force.

$-\rho \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ is called the (again fictitious) Centrifugal Force.
6. We try to write the centrifugal force as the gradient of something, given that the density is uniform. The result as stated will be true providing we can show that
\[ \rho \Omega \times (\Omega \times \mathbf{r}) = -\nabla (\rho \left[ \Omega \times \left[ \Omega \times \mathbf{r} \right] \right] / 2) \]

Since the density is uniform it cancels out; the LHS \( \Omega \times (\Omega \times \mathbf{r}) \) is then \( \Omega \times \left( \Omega \times \mathbf{r} \right) = \Omega \times \mathbf{r} \).

The RHS has i-th component
\[ -\frac{1}{2} \left\{ \varepsilon_{jnp} \varepsilon_{mqr} \left( \varepsilon_{jmn} \Omega_i \right) \right\} / 2 \]

(note the use of different dummy suffixes in the two versions of \( \Omega \times \mathbf{r} \))

\[ = -\frac{1}{2} \left( S_{pq} S_{mr} - S_{pm} S_{qr} \right) \frac{\partial p \Omega_i}{\partial x_i} \left( \frac{\partial q \Omega_m + \partial m \Omega_q}{\partial x_i} \right) \]

on using the product rule and remembering \( \Omega \) is constant.

But \( \frac{\partial q \Omega_m}{\partial x_i} = \partial m \Omega_q \) etc.

So this is
\[ -\frac{1}{2} \left( S_{pq} S_{mr} - S_{pm} S_{qr} \right) \frac{\partial p \Omega_i}{\partial x_i} \left( \frac{\partial q \Omega_m + \partial m \Omega_q}{\partial x_i} \right) \]

Going for broke with the substitution property of the \( S \)'s, this is
\[ -\frac{1}{2} \partial_m \partial_p \partial_i \dot{r} + \frac{1}{2} \partial_q \partial_\Omega \partial_i \dot{r} - \frac{1}{2} \partial_\Omega \partial_p \partial_i \dot{r} + \frac{1}{2} \partial_\Omega \partial_\Omega \partial_i \dot{r} \]

But \( \partial_\Omega \partial_\Omega = \nabla^2 \), and \( \partial_q \partial_\Omega = \partial_\Omega \partial_q \) are constants.

So this is the i-th component of
\[ -\frac{1}{2} \nabla^2 \dot{r} + (\dot{r} \cdot \nabla) \dot{r} \], and so LHS = RHS, and we can write \( -\nabla p - \rho \left[ \Omega \times (\Omega \times \mathbf{r}) \right] \) as \( -\nabla p \),
where \( \mathbf{p} = \rho - \rho \left[ \Omega \times (\Omega \times \mathbf{r}) \right] / 2. \)
7. For steady flow with viscosity negligible, the equation in Q5 becomes
\[ p \left( u \cdot \nabla u + 2 \Omega \times u \right) = -\nabla \mathcal{P} \]
where we have taken the body force to be zero, used the modified pressure featured in Q6, and dropped suffixes RF on the understanding that everything is now treated in the rotating frame. If the flow is slow in this frame, with some characteristic velocity \( U \) and length scale \( L \), the \( u \cdot \nabla u \) term will be negligible if \( U^2 \ll 2L \mathcal{U} \), i.e., if \( \mathcal{R} \omega = \frac{U}{2L \mathcal{U}} \ll 1 \). (This non-dimensional number is known as the Rossby Number.) Then, 
\[ 2 \Omega \times u = -\nabla \mathcal{P} \]

[In passing, note that \( u \cdot (\Omega \times u) = 0 \) [scalar triple product] \( \Rightarrow u \cdot \nabla \mathcal{P} = 0 \) \( \Rightarrow \) the pressure gradient has no component in the direction of \( u \Rightarrow u \) flows along lines of constant pressure. This explains why on weather maps the wind always flows approximately parallel to isobars, an otherwise counter-intuitive fact.]

Now take the curl, which gives 
\[ \nabla \times (\Omega \times u) = 0 \] 
\[ \Rightarrow \text{(vector identities, p123)} \] 
\[ u \cdot \nabla \mathcal{P} = \frac{\partial}{\partial x} \mathcal{P} + \frac{\partial}{\partial y} \mathcal{P} + \frac{\partial}{\partial z} \mathcal{P} = 0 \]
But \( \Omega \) is constant, and incompressibility \( \Rightarrow \nabla \cdot u = 0 \). So \( (\Omega \cdot \nabla) u = 0 \) - no component of \( u \) can have any variation in the direction of \( \Omega \). (Taylor-Proudman theorem).
8. \[ \frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa \nabla^2 T. \]

Write \[ u = U \hat{u}, \quad x = L \hat{x}, \quad t = \left( \frac{L}{U} \right) \hat{t} \]

Then \[ \left( \frac{U}{L} \right) \frac{\partial T}{\partial \hat{t}} + U \hat{u} \cdot \hat{\nabla} T = \kappa \hat{\nabla}^2 \hat{T} \]

when the scalings are carried out.

\[ \therefore \frac{\partial T}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} T = \left( \frac{\kappa}{UL} \right) \hat{\nabla}^2 T = \frac{1}{Pe} \hat{\nabla}^2 T. \]

If \( Pe \) is large, the RHS is negligible, at least over most of the region, so we expect \[ \frac{\partial T}{\partial \hat{t}} \approx 0, \] ie \( T \) is constant following the flow. However, this has reduced the order of the system; thus for most boundary conditions we anticipate boundary layers in the \( T \)-field, of thickness \( S \approx L \), small enough to make the RHS locally of order 1, ie \[ \frac{1}{Pe} \] has to be of order 1. This requires \[ S \approx (Pe)^{-\frac{1}{2}} L, \] ie we expect boundary layers of thickness \( (Pe)^{-\frac{1}{2}} \times L \), enclosing regions with \( \frac{\partial T}{\partial \hat{t}} \approx 0 \). This is found to happen in numerical simulations of thermal convection.
9. With \( \mathbf{u} = (u(y), 0, 0) \) and an applied pressure gradient \( \frac{dp}{dx} = -G \), the \( x \)-component of the Navier-Stokes equation is (see lectures)

\[
\mu \frac{d^2u}{dy^2} = -G.
\]

Integrating,
\[
u = \frac{G}{2\mu} y^2 + Ay + B.
\]

The BC's are: \( u = 0 \) at \( y = 0 \), \( u = U \) at \( y = d \).

Thus \( B = 0 \) and \( A = \frac{U}{d} - \frac{Gd}{2\mu} \).

This gives the solution for the velocity

\[
u = \frac{U}{d} y + \frac{G}{2\mu} y(y-d).
\]

This agrees with \( G = 0 \) result.

The tangential stress at any level is \( \mu \frac{du}{dy} \)

\[
= \left[ \frac{U}{d} - \frac{Gd}{2\mu} + \frac{Gy}{\mu} \right] \mu.
\]

At \( y = 0 \) this is \( \frac{U}{d} - \frac{Gd}{2} \) acting in the positive \( x \)-direction.

At \( y = d \), this is \( \frac{U}{d} + \frac{Gd}{2} \) acting in the negative \( x \)-direction (negative since the normal points in the opposite direction; the force has to oppose both the boundary motion and the pressure gradient).
Our only hope of finding an easy solution to this problem is to suppose that the flow is uniform on each nested ellipse in the family \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda, \quad 0 \leq \lambda \leq 1 \). So try

\[ u = \left( 0, 0, U(\lambda) \right) \]

and pray it works....

The boundary conditions to be satisfied are that \( U = 0 \) when \( \lambda = 1 \) (no-slip condition), with \( U \) and \( \frac{dU}{d\lambda} \) both finite at \( \lambda = 0 \).

Work in Cartesian coordinates: then

\[ \nabla^2 u = \left( 0, 0, \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \]

With the flow as hypothesised, \( \nabla \cdot \mathbf{u} = 0 \) (unidirectional flow).

So \( \frac{\partial p}{\partial x} \) and \( \frac{\partial p}{\partial y} \) are zero from the \( x \) and \( y \) amplitudes of Navier-Stokes, and the \( z \)-component gives

\[ -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = 0, \quad \Rightarrow \frac{\partial p}{\partial z} = \text{const}, \]

= \(-G\) say, as in the circular case.

Now \( \frac{\partial U}{\partial x} = \frac{dU}{d\lambda} \frac{d\lambda}{dx} = \frac{2x}{a^2} \frac{dU}{d\lambda}, \quad \frac{\partial^2 U}{\partial x^2} = \frac{2}{a^2} \frac{dU}{d\lambda} + \frac{4x^2}{a^4} \frac{d^2U}{d\lambda^2} \)

Similarly \( \frac{\partial^2 U}{\partial y^2} = \frac{2}{b^2} \frac{dU}{d\lambda} + \frac{4y^2}{b^4} \frac{d^2U}{d\lambda^2} \)

Hence

\[ \mu \left[ 4 \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right) + 2 \left\{ \frac{1}{a^2} + \frac{1}{b^2} \right\} \frac{dU}{d\lambda} \right] = -G. \]
At this stage the presence of the coefficient of \( \frac{d^2U}{d\lambda^2} \) is not a simple function of \( \lambda \) and things look pretty hopeless. However, there is one simple thing yet to try: we can suppose \( \frac{d^2U}{d\lambda^2} = 0 \) and see if this works.

Then \( U = A \lambda + B \), and if we take
\[
2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right) A = -\frac{G}{\mu},
\]
this will solve the equation, and have \( U, \frac{\partial U}{\partial \lambda} \) finite at \( \lambda = 0 \).

To satisfy the no-slip condition at \( \lambda = 1 \), we need \( B = -A \). This leads to the solution
\[
U = \frac{G a^2 b^2}{2 \mu} \left[ 1 - \left( \frac{x^2 + y^2}{a^2 + b^2} \right) \right],
\]
which satisfies the equation and boundary conditions.

It also reduces to the Poiseuille flow result \( U = G (a^2 \lambda) \) when \( a = b \). [Solution by spiritualism.]

11. Adopt coordinates \( x \) and \( y \) along and perpendicular to the plane, as shown. Assume \( u = (u(y), 0) \) in these coordinates. Then \( u \cdot \nabla u \equiv 0 \), and the equations are similar to the Couette flow.
problem except that there is a body force \( \mathbf{g} \) per unit mass, with components \((g \sin \alpha, -g \cos \alpha)\). Thus the equations are

\[
\begin{align*}
\frac{\partial \rho}{\partial x} &= \rho g \sin \alpha + \mu \frac{d^2 u}{dy^2} \\
0 &= -\rho \frac{\partial p}{\partial y} - \rho g \cos \alpha.
\end{align*}
\]

We assume there is no applied pressure gradient in the \( x \)-direction, so \( \frac{\partial p}{\partial x} = 0 \); the second equation just tells us how the gravity forces a pressure gradient \( 1^r \) to the plane (so that \( p(y) = p_0 + \rho g \sin \alpha (h-y) \)) but is otherwise uninteresting. Solving the first equation gives

\[
u = -\frac{\rho g \sin \alpha}{2\mu} y^2 + Ay + B.
\]

The BC's are: \( u = 0 \) at \( y = 0 \) \( \Rightarrow B = 0 \)

surface stress \( \mu \frac{\partial u}{\partial y} = 0 \) at \( y = h \) \( \Rightarrow A = \frac{\rho gh \sin \alpha}{\mu} \)

Thus the velocity is

\[
u = \frac{\rho g \sin \alpha}{2\mu} \left\{ 2hy - y^2 \right\}
\]

The surface stress tangentially on the plane \( y = 0 \) is \( \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \rho g h \sin \alpha \). This is the force exerted by the fluid on the plane; thus the plane must exert a force \( -\rho g h \sin \alpha \) unit area (up the plane) on the fluid. But this is equal and opposite to the component down the plane of the weight of unit area of fluid. \( \therefore \) The forces are in balance.
12. We use coordinates \((R, \phi, z)\) with \(\phi\) azimuthal, \(z = 0\), and \(z\) pointing down the axis of the pipe. As before, suppose \(u = (0, 0, U(R))\), so that \(\nabla \cdot u = 0\). As before, the only difference gravity makes in the streamwise \((z)\) direction is to add a body force \(pg \sin \alpha\) per unit volume; thus the Poiseuille - flow equation is modified to

\[-\frac{\partial p}{\partial z} + pg \sin \alpha + \mu \left( \frac{d^2 u}{dR^2} + \frac{1}{R} \frac{du}{dR} \right) = 0.\]

Thus the solution is formally identical to the standard case if we modify the applied pressure gradient \(\partial p/\partial z\) by using \(G + pg \sin \alpha\) in place of \(G\).

Hence the solution for the velocity is

\[U = \frac{(G + pg \sin \alpha)}{4\mu} (a^2 - R^2).\]

To check the force balance, suppose \(G\) itself is zero.

The tangential stress (unit area) is \(\mu \partial U/\partial R\) (at the pipe wall \(R=a\))

\[\text{Sign is opposite to last question because the outward-pointing normal is pointing in the opposite direction. The simplest way to get the direction of the force is to use common sense.}

The wall must obviously exert an upward force \(pga \sin \alpha\) per unit area on the fluid.

The force on the fluid in unit length of pipe is

\[2\pi a \cdot pg a \sin \alpha = \left(2\pi a^2 pg \right) \sin \alpha.\]

This is numerically equal to the weight component (down the slope) of the fluid. So once again the forces balance.
13. As in the impulsively started plate problem, the velocity is taken to be \( u(y,t), 0, 0 \), and \( u \) satisfies the diffusion equation
\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial t}, \text{ with } u = V \cos \omega t \quad (y=0) \\
\lim_{y \to \infty} u = 0.
\]
We assume the initial condition is irrelevant in that any transients have already decayed by viscous dissipation; thus we examine only the forced response. 

Try a separable solution \( u = \text{real part of } f(y) e^{i\omega t} \)

Then \( f'' = i\omega f \) and \( f \) must satisfy \( f = V(y=0) \)
\[
f'' - \frac{i\omega}{\lambda} f = 0 \quad ; \quad \text{solutions of this are } f = e^{\pm \sqrt{\frac{i\omega}{\lambda} y}}. \text{ Now } \sqrt{i} = e^{i\pi/4} = \frac{1}{\sqrt{2}} (1+i).
\]

The G.S. for \( f \) is \( f = Ae^{\lambda(1+i)y} + Be^{-\lambda(1+i)y} \), where \( \lambda = \sqrt{\frac{\omega}{\sqrt{2\upsilon}}} \). For \( y \to \infty \), \( f \to 0 \), so must have \( \lambda \) zero; the other B.C. gives \( B = V \).

Hence \[ u = \text{Re} \left\{ V e^{-\lambda y} e^{i(\omega t - \lambda y)} \right\} \]
\[
\text{ie } u = V e^{-\lambda y} \cos(\omega t - \lambda y)
\]

Thus the velocity decays in magnitude away from the plane, being reduced by \( \frac{1}{e} \) in a distance \( \frac{1}{\lambda} = \frac{2\upsilon}{\omega} \).

Thus high-frequency disturbances travel hardly any distance at all, low frequency go further in. The phase of the wave varies with height, requiring \( y = \frac{2\pi}{\omega} = \frac{2\pi}{\omega} \sqrt{\frac{2\upsilon}{\omega}} \) to be once again in phase with the movement of the plate.
14. As in lectures, the equation is $\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$,

but now the BC's are $\mu \frac{\partial u}{\partial y} = -S$ at $y = 0$,

$u \to 0$ as $y \to \infty$.

Try a similarity solution $u = \frac{S}{\mu} y \cdot F(\eta)$

where $\eta = \frac{y}{\sqrt{4 \mu t}}$. Then $\frac{\partial u}{\partial t} = \frac{S y}{\mu} F'(\eta) \left\{-\frac{y}{2\sqrt{4\mu t} \cdot t}\right\}$

$\frac{\partial u}{\partial y} = \frac{S}{\mu} \left[F(\eta) + y F'(\eta)\right]$.

$\frac{\partial^2 u}{\partial y^2} = \frac{S}{\mu} \left[\frac{2}{\sqrt{4\mu t}} F'(\eta) + y F''(\eta)\right]$.

Hence the substitution of these into the diffusion eqn gives

$-\frac{y^2}{2\sqrt{4\mu t} \cdot t^{\frac{3}{2}}} F'(\eta) = \frac{2 \mu}{\sqrt{4\mu t}} F'(\eta) + y \mu F''(\eta)$.

Multiplying by $\frac{4\mu t}{y}$ gives $F''(\eta) + F'(\eta) \left\{\frac{2}{\eta} + 2\eta^2\right\} = 0$.

Treating this as a first-order integrating-factor-type DE for $F'$ gives $\eta^2 e^{\eta^2} F'(\eta) = C$.

ie $F'(\eta) = \frac{C e^{-\eta^2}}{\eta^2}$.

Thus formally $F = C \int \frac{\eta^2 e^{-\eta^2}}{\eta^2} d\eta + D$.

will give the solution. Given that we want $F \to 0$ as $\eta \to \infty$ we can eliminate $D$ by taking as the solution $F = -C \int_{\eta}^{\infty} e^{-\eta^2} d\eta$. 

To relate \( C \) to the \( y = 0 \) BC, we need
\[
\frac{\partial u}{\partial y} = -S \quad \text{at} \quad y = 0,
\]
\[\text{ie}\]
\[
SF(\eta) + SyF'(\eta) \frac{1}{\sqrt{4\pi t}} = -S,
\]
\[\text{ie}\]
\[
F(\eta) + \eta F'(\eta) = -1 \quad \text{at} \quad \eta = 0.
\]
This is the BC to be applied. Using the \( F \) we have found, this enables \( C \) to be determined.

For the interested only: (much harder than expected cause standard)
From the singular-looking nature of the integral, it is not obvious that this will actually work, especially as \( \eta F'(\eta) \)
\[\frac{Ce^{-\eta^2}}{\eta}\]
as \( \eta \to 0 \). However, it does: near \( \eta = 0 \)
we can asymptotically expand the integral for \( F \) as
\[
F = + \frac{Ce^{-\eta^2}}{\eta} \left[ \int \frac{2\eta e^{-\eta^2}}{\eta} d\eta \right] \quad \text{using integration by parts.}
\]
Thus \( F \) is actually
\[
\frac{-Ce^{-\eta^2} + 2C \frac{\sqrt{\pi}}{2} \left( 1 - \text{erf}(\eta) \right)}{\eta}
\]
as \( \eta \to 0 \).
Now it can be seen that the infinite bits in \( \bigstar \) as \( \eta \to 0 \) cancel each other out. Thus \( \bigstar \) gives, as \( \eta \to 0 \)
and using \( \text{erf}(0) = 0 \), \[2C \frac{\sqrt{\pi}}{2} = -1, \quad \text{ie} \quad C = -\frac{1}{\sqrt{\pi}}\]
Using \( y = \sqrt{4\pi t} \eta \) then gives the full solution \( S \cdot y \cdot F \)
as
\[
\frac{u}{\mu} = \frac{S}{\mu} \frac{1}{\sqrt{\pi t}} \left[ \int e^{-\frac{y^2}{4\pi t} - \frac{y}{\sqrt{4\pi t}}} + \text{erf} \left( \frac{y}{\sqrt{4\pi t}} \right) \right]
\]
This is finite for finite \( y \), \( t \) (and zero at \( t = 0 \) ).
15. For this problem \( u \), \( \frac{\partial u}{\partial t} \) is still zero and the only important equation to be solved is that describing the \( x \)-velocity. This is

\[
\frac{\partial u}{\partial t} = -\frac{G}{\rho} + \nu \frac{\partial^2 u}{\partial y^2},
\]

the diffusion equation with a forcing term.

This is to be solved subject to the BC's \( u = 0 \) at \( y = \pm a \), and the initial condition \( u = 0 \) at \( t = 0 \).

The steady solution to which the flow will evolve is the above with \( \frac{\partial u}{\partial t} \) set equal to zero; the solution satisfying \( u = 0 \) at \( y = \pm a \) is \( u = u_1 = \frac{G}{2\mu} (a^2 - y^2) \).

As suggested, put \( u = u_1(y) + U(y,t) \). Then

\[
\frac{\partial U}{\partial t} = \frac{G}{\rho} + 2\nu \frac{\partial^2 u_1}{\partial y^2} + \nu \frac{\partial^2 U}{\partial y^2} = \nu \frac{\partial^2 U}{\partial y^2}
\]

since \( u_1 \) satisfies the time-independent problem.

Thus the problem for \( U \) is to solve

\[
\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial y^2}
\]

with BC's \( U = 0 \) at \( y = \pm a \)

and \( U = -u_1 = -\frac{G}{2\mu} (a^2 - y^2) \) at \( t = 0 \).

Try \( U = Y(y)T(t) \).

Then

\[
\frac{T}{\nu T} = \frac{Y''}{Y} = \text{some constant} = -k^2 \text{say.}
\]

(in anticipation that we won't be able to satisfy both BC's unless the constant is negative.)

Hence \( Y \propto \sin ky \) or \( \cos ky \), and \( T \propto e^{-k^2 \nu t} \).
so that a separable solution is

\[ e^{-kt}u = (A \sin ky + B \cos ky). \]

Now \( U = 0 \) at \( y = \pm a \) \( \Rightarrow \pm A \sin ka + B \cos ka = 0 \)

Thus we must have \( B \cos ka = A \sin ka = 0 \). Either \( A \) or \( B \) must be zero; since the initial condition is symmetric about \( y = 0 \), we take cosines only, i.e., put \( A = 0 \). Then \( \cos ka = 0 \) \( \Rightarrow k = \left( n + \frac{1}{2} \right) \pi \)

for \( n = 0, 1, 2, \ldots \)

Thus the general solution is

\[ U(y, t) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{(2n+1)\pi y}{a}\right) e^{-k^2 \pi^2 t} \]

where \( k_n = \left( n + \frac{1}{2} \right) \pi \).

At \( t = 0 \), this has to equal \( -\frac{G}{2\mu} (a^2 - y^2) \).

Thus

\[ -\frac{G}{2\mu} (a^2 - y^2) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{(2n+1)\pi y}{a}\right) \]

Hence

\[ B_n = -\frac{G}{2\mu a} \int_{y=0}^{a} (a^2 - y^2) \cos\left(\frac{(2n+1)\pi y}{a}\right) dy \ (n \neq 0) \]

\[ = -\frac{G}{\mu a} \left[ \left( \frac{a^2 \sin \left( \frac{(2n+1)\pi y}{a} \right)}{2n+1} \right) \right]_0^a + \left[ \frac{2y a^2}{(2n+1)^2 \pi^2} \sin \left( \frac{(2n+1)\pi y}{a} \right) \right]_0^a \]

\[ = \frac{G}{\mu a} \cdot \frac{4a}{(2n+1) \pi^2} \left[ \frac{2a^2}{2a} \left( \frac{a^2 \sin \left( \frac{(2n+1)\pi y}{a} \right)}{2n+1} \right) \right]_0^a - \frac{4a^2}{(2n+1)^2 \pi^2} \frac{\sin \left( \frac{(2n+1)\pi y}{a} \right)}{2a} \]

\[ = \frac{G}{\mu a} \cdot \frac{16a^2}{(2n+1)^3 \pi^3} \]

\[ = \frac{G}{\mu a} \cdot \frac{(2n+1)^3 \pi^3}{16a^2} \]

\[ n = 0 \]

The slowest decaying term in the series is \( n = 0 \), with characteristic decay time \( T = \frac{4a^2}{\mu \pi^2} \), of order the diffusion timescale \( \frac{a^2}{\mu} \).

This estimates the time to reach the steady state.