Potential Flows, Airfoil Theory

1. Given a complex potential \( W(z) = \phi(x, y) + i\psi(x, y) \), use the Cauchy-Riemann equations to show that the curves \( \phi = \text{constant} \) and \( \psi = \text{constant} \) intersect at right angles. Show also that the curves \( \phi = \text{constant} \) give the streamlines of some other 2-D flow, and identify what this flow is when \( w(z) = A\log z \) and when \( w(z) = A/z \), \( A \) being a real constant.

2. Two line vortices each of circulation \( \kappa \) are situated on the real axis at \( z = \pm a \). A third vortex of strength \( -\kappa/2 \) is at the origin. Using complex variable methods, show that each vortex is at rest, and sketch the streamlines of the flow.

3. A stream of speed \( U \) parallel to the \( x \)-axis moves above the plane \( y = 0 \) and there is a line dipole of strength \( m \) situated at \( z = ai \), inclined at \( \alpha \) to the \( x \)-axis. Show that if \( U = 2m\cos\alpha/a^2 \) there is a stagnation point at the origin.

4. Suppose a 2-D incompressible flow is irrotational except at isolated singularities, and takes place with complex potential \( f(z) \). All singularities lie outside the region \( |z| = a \). Show that \( w(z) = f(z) + \overline{f(a^2/z)} \), where an overbar denotes complex conjugation, gives a suitable complex potential for the same flow with the added insertion of a cylindrical boundary \( |z| = a \), the flow being understood to take place in the region outside the boundary. (This is the Milne-Thomson Circle Theorem: to demonstrate it you should show (i) that the flow has the same behaviour for large \( z \); (ii) that the circle \( |z| = a \) is a streamline with \( \psi = \text{constant} \), and (iii) that any additional singularities introduced by the new term lie within \( |z| = a \), so that \( w(z) \) is analytic with real part satisfying \( \nabla^2 \phi = 0 \) except at the location of the original singularities.)
5. A line vortex of strength $\kappa$ is situated a distance $b > a$ from the centre of a cylinder of radius $a$; the flow satisfies the conditions for the theorem in part (a) to be valid. Denoting the instantaneous location of the vortex by $z_0$, use the Circle Theorem to write down the complex potential for this flow, and show that up to the addition of an arbitrary (physically irrelevant) constant it may be written

$$w(z) = -\frac{i\kappa}{2\pi} \left( \log(z - z_0) + \log(z - a^2/z_0) \right).$$

Work out the complex velocity at the location of the original vortex, excluding the self-induced component in the potential. Hence show that the vortex moves around the cylinder in a circle of radius $b$ with angular velocity $\Omega$ given by

$$\Omega = -\frac{\kappa a^2}{2\pi b^2 (b^2 - a^2)}.$$

6. Consider the flow domain in the $z$-plane given by the infinite strip $0 < y < a$. Describe what domain this maps to in the $\zeta$-plane, for the conformal mapping given by $\zeta(z) = \exp(\pi z/a)$ (identify carefully what the boundaries map into). Write down the complex potentials for the flows in the $\zeta$-plane which correspond to the two uniform streams $w(z) = \pm U z$ in the strip in the $z$-plane, and say what they represent. Which of these $\zeta$-flows would you expect to model a real-life flow (i.e. one that can accommodate the no-slip condition via a boundary layer), and why? (Consider the build-up of pressures and counterflows in the boundary layer.)

7. Consider the same $z$-flow domain as in the last question, but restricted now to its right half, so that the strip can be regarded as divided by a barrier $x = 0$. It is desired to find the flow due to a line source of strength $S$ placed at $z_0 = x_0 + iy_0$ within this domain. Verify that the conformal mapping $\zeta = \cosh(\pi z/a)$ maps the flow domain into the upper half plane, and use an image method to write down the solution in the $\zeta$-plane. Hence deduce that the complex potential for the original flow in the $z$-plane is

$$w(z) = (S/2\pi) \log\left\{ \frac{[\cosh(\pi z/a) - \cosh(\pi z_0/a)][\cosh(\pi z/a) - \cosh(\pi \bar{z}_0/a)]}{[\cosh(\pi z/a) - \cosh(\pi z_0/a)][\cosh(\pi z/a) - \cosh(\pi \bar{z}_0/a)]} \right\}.$$

8. Consider the same flow domain as in the last question, ie the semi-infinite strip bounded by the lines $x = 0, y = 0$ and $y = a$. Suppose fluid flows in at a rate $2\pi S$ at the bottom corner $x = 0, y = 0$, and leaves at the same rate at the top corner $x = 0, y = a$. Show that the complex potential for the resulting flow is

$$w(z) = 4S \log[\tanh(\pi z/2a)],$$

and show that half the stream lies between $x = 0$ and the streamline which cuts $y = a/2$ at the point $x = (a/\pi) \log(1 + \sqrt{2})$. 
9. The following is intended to model a wind blowing over a wall of height \( h \). Show that the mapping \( \zeta = h\sqrt{z^2 - 1} \), where \( z = x + iy, \zeta = \xi + i\eta \), sends the real \( z \)-axis onto the real \( \zeta \)-axis together with the line segment \( \zeta = i\eta \) \((0 \leq \eta \leq 1)\) taken twice over, with the two upper half planes corresponding. Deduce the complex potential for the flow of a stream uniform and parallel to the \( \xi \)-axis far from the origin, as it passes over the wall. Calculate the corresponding fluid speed, noting any unphysical features, and find the pressure distribution over the front and back of the wall. To what extent do you think the flow will be like this in practice? (Find a wall and try it out; now it is autumn, leaves accumulated behind the wall may swirl around and help you visualise the flow.)

10. The following is an attempt to model the wind blowing past the corner of a building. Let the latter be given in the \( \zeta \)-plane by the boundaries \( \xi = 0, \eta \leq 0 \) and \( \eta = 0, \xi \geq 0 \). A stream which is directed up the wall as \( \eta \to -\infty \) blows along the building from south to north. The flow separates as it goes round the corner; the resulting perturbation is modelled as a line vortex at \( \zeta = \zeta_0 \), with \( \Re(\zeta_0) > 0 \) (again, this can be observed in the city with the help of autumn leaves!) Use the transformation \( \zeta = z^{3/2} \) and an easier flow in the \( z \)-plane to give the complex potential for this model.

11. The force element \((dX, dY)\) acting at \((x, y)\) has a moment \( dM = (xdY - ydX) \). Show that this can be written \( dM = \Re\{iz(dX - idY)\} \) and hence use manipulations similar to those involved in deriving Blasius’ theorem to show that the total couple acting on the body is

\[
M = -\frac{1}{2}\rho\Re\oint_C z\left(\frac{dw}{dz}\right)^2dz.
\]

12. Use Blasius’ theorem to verify that the force on a cylinder in a stream of magnitude \( U \) with a circulation \(-\kappa\), whose complex potential is (cf lectures)

\[
w(z) = U(z + \frac{a^2}{z}) + i\frac{\kappa}{2\pi}\log z,
\]

consists solely of a lift \( \rho U \kappa \). Verify also that there is no couple acting on the cylinder.

13. Points on the cambered Zhukovski airfoil have coordinates given by

\[
\zeta \equiv \xi + i\eta = z + \frac{a^2}{z}
\]

where \( z = -\delta + i\epsilon + be^{i\theta} \), \( \theta \) runs from 0 to \( 2\pi \) going round the airfoil boundary, and \( b, \delta, \epsilon \) are chosen with \( b^2 = (a + \delta)^2 + \epsilon^2 \), so that the circle in the \( z \)-plane goes through \( z = a \). Calculate \( \xi(\theta), \eta(\theta) \) to first order in \( \delta, \epsilon, \) assuming these are small. Calculate the extreme values of \( \xi \), and hence the chord of the airfoil (distance from leading to trailing edge), to first order. Estimate roughly the extreme values of \( \eta \). Hence sketch the airfoil; what is its approximate maximum thickness?
14. Show that the limit as $\delta \to 0$ of the airfoil specified in the last question is a circular arc passing through the points $\zeta = 2a, \zeta = -2a$. (Note: this is quite hard. The easiest method, according to Lighthill, is to show that for any $\zeta$ on the airfoil the angle between the lines joining that point to the points $\zeta = \pm 2a$ takes the constant value $2 \tan^{-1}(a/\epsilon)$. Batchelor (pp446-447) has an elegant solution.)

15. (i) An inviscid two-dimensional stagnation point flow has $x$ and $y$ components $u = \alpha x$ and $v = -\alpha y$. Find expressions for the velocity potential $\Phi(x, y)$ and the stream function $\psi(x, y)$. Use the latter to give a sketch of the streamlines. Assuming that the flow is restricted to the region above a flat plate $y = 0$, what boundary condition is satisfied at this plate? Using Bernoulli, write down an expression for the pressure $p$, taking $p = p_0$ at the origin.

(ii) Now consider the above flow in the case where the fluid is viscous. What are the relevant boundary conditions at the plate now? Introduce a stretched variable $\eta = (\alpha/\nu)^{1/2} y$, where $\nu$ is the kinematic viscosity. Seek a solution of the form

$$u = \alpha x f'(\eta), \quad v = -(\nu \alpha)^{1/2} f(\eta),$$

where $f(\eta)$ is a function to be determined. In line with what you found in (i), take $p = p_0 - \rho \alpha^2 x^2/2 + g(\eta)$, where $g(\eta)$ is also to be determined. Use the $x$-component of the steady Navier-Stokes equation to show that $f(\eta)$ satisfies the exact equation

$$f''' + ff'' + 1 - (f')^2 = 0.$$ 

Use the $y$-component (which can be integrated once) to show that

$$g = -\rho \nu \alpha \left( f' + \frac{1}{2} f'^2 \right).$$

What boundary conditions should be imposed on the first equation to ensure that it provides a boundary layer solution matching to the flow specified in (i)?