The University of Sydney

Mathematics III

Fluid Dynamics 2012

Example Sheet 4

Miscellaneous problems, harder and non-examinable.

1. For an inviscid fluid with no body forces, derive the momentum theorem

$$\frac{d}{dt}\int_V \rho u_i dV = -\oint_S (p\delta_{ij} + \rho u_i u_j)dS_j$$

for a fixed volume V with surface S.

2. A steady two-dimensional flow consists of two identical jets of inviscid, homogeneous fluid meeting symmetrically at an angle 2β; the jets are embedded in air of uniform pressure. The liquid flows away from the impact region along the plane of symmetry, as two more oppositely directed jets. Use the previous question and Bernoulli's equation to show that the rates of volume flow in the two exit jets are in the ratio

$$1 + \cos\beta : 1 - \cos\beta\,.$$

(Hint: Assume the flow in each jet is unidirectional far from the impact region, and that the flow is irrotational. Don't try to find the detailed velocity field in the impact region.)

(This idea is the basis for an explosive device known as a shaped charge, used to blow holes in tank armour.)

3. Show by conformal mapping that the path of a line vortex in the region between two intersecting straight planes is given by

$$r\sin n\theta = \text{constant}\,,$$

where (r, θ) are polar coordinates and $\theta = 0$ and $\theta = \pi/n$ are the two boundaries.

4. Line vortices of equal strength κ are equidistant on an infinite straight line, each separated from its neighbour by a distance a. A similar parallel row of opposite vortices with strengths $-\kappa$ lies a perpendicular distance b from the first. Show that the speed at which all the vortices move along the rows is

$$\frac{\kappa}{2a}\coth\frac{\pi b}{a}$$

if a vortex in one row is directly opposite one in the other row, or

$$\frac{\kappa}{2a}\tanh\frac{\pi b}{a}$$

if the rows are symmetrically staggered. (This type of model is called a Karman vortex street; for certain ranges of Reynolds number it can represent the wake behind an object in a stream. The question is hard but interesting...)

5. Consider the stability of the plane-parallel shear flow $U(y) = 1$ $(y \geq 1)$, $= y$ $(-1 \leq y \leq 1)$, $= -1$ $(y \leq -1)$. The fluid is inviscid, and the velocity perturbation may be taken as obeying Rayleigh's equation. Write down the form of the solution in the three regions $y > 1$, $|y| < 1$, $y < -1$, taking into account the boundary conditions that the velocity perturbation should decay as $|y| \to \infty$.

The perturbed stream function must be continuous at $y = \pm 1$, giving that $f(y)$ is continuous there (where $f(y)$ is defined as in lectures). $U''(y)$ is discontinuous at $y = \pm 1$; however, show that Rayleigh's equation may also be written

$$\frac{d}{dy}\Big((U-c)f' - fU'\Big) - k^2(U-c)f = 0\,,$$

then integrate this from just below $y = \pm 1$ to just above to show that $(U-c)f' - fU'$ must be continuous at these points. Use these four patching conditions to show that for a non-zero solution

$$c^2 = \frac{1}{4k^2}\Big((1-2k)^2 - e^{-4k}\Big)\,.$$

Deduce that the flow is unstable for $0 < k < k_s$, where k_s solves $1 - 2k + e^{-2k} = 0$. Find an equation for the wavenumber k_m which maximises the growth rate kc_i; show that the solution for k_m is of order 1, and comment on the relation of this to the "ultra-violet catastrophe" in the analysis of the Kelvin-Helmholtz instability.

6. Two static layers of fluid lie one on top of the other, with gravity acting. The upper fluid is in $y > 0$ and has density ρ_u; the lower is in $y < 0$ and has density ρ_ℓ. Show that there is a hydrostatic equilibrium solution with the weight balanced by pressure gradients, even if the top fluid is heavier than the lower.

Now perturb the interface $y = 0$ so that it is given by $y = \zeta = \zeta_0 e^{i(kx-\omega t)}$. Assuming inviscid, incompressible flow (so that the velocity potential in the two regions satisfies Laplace's equation), write down the form of this potential in the two regions, assuming that the disturbance decays as $|y| \to \infty$. Treating the boundary in the same way as instructed in Assignment 3 Q.4, derive equations connecting ζ_0 and the amplitudes of the velocity potentials in the upper and lower regions. Hence derive the dispersion relation

$$\omega^2 = gk\frac{\rho_\ell - \rho_u}{\rho_\ell + \rho_u}\,.$$

Deduce that (reassuringly!) the flow is unstable if $\rho_\ell < \rho_u$, with shortest wavelengths being the most unstable (this is called Rayleigh-Taylor instability). What happens in the opposite case, in particular if $\rho_u = 0$?