The University of Sydney
Faculties of Arts, Economics, Education, Engineering and Science
Math2004/2904
Lagrangian Dynamics/Lagrangian Dynamics (Advanced)

November 2004

Time Allowed: Two Hours

Lecturer: L. Poladian

This Examination consists of 4 pages numbered from 1 to 4. There are 5 questions numbered from 1 to 5.

Questions are of equal value.
The number of marks awarded for each subsection of each question is shown.

Students enrolled in MATH-2004 should attempt Questions 1 to 4 and may attempt Question 5.

Students enrolled in MATH-2904 should attempt all questions.
1. The position of a point on the surface of a circular cylinder of radius $a$ and axis along the $z$-axis is represented by the cylindrical polar coordinates $(r, \theta, z)$.

(i) (3 marks)
State the formulae for the cartesian coordinates $(x, y, z)$ in terms of these cylindrical coordinates.

(ii) (4 marks)
Let $ds$ be the small distance between neighbouring points. Show that in cylindrical coordinates
$$ ds^2 = dr^2 + r^2 d\theta^2 + dz^2. $$

(iii) (3 marks)
Find the expression for the path length on the surface of the cylinder with constant $r = a$
$$ I = \int_{z_0}^{z_1} \frac{ds}{dz} dz. $$

(iv) (4 marks)
Find the function $\theta(z)$ that describes the path of least length between the points $(a, 0, 0)$ and $(a, \pi, H)$.

2. A particle of unit mass is projected around the inner surface of an upright circular cone with its vertex down. The equation of the conical surface is $x^2 + y^2 = \frac{z^2}{a^2}$.

(i) (4 marks)
Find the Lagrangian for the particle in terms of the cylindrical coordinates $r$ and $\theta$.

(ii) (4 marks)
Find two integrals of the motion.

(iii) (3 marks)
If the angular momentum is $h = \sqrt{2ga}$ and the energy is $E = 2ga$, show that
$$ (1 + a^2)r^2 + 2ga \left( r - 2 + \frac{1}{r^2} \right) = 0. $$

(iv) (3 marks)
Using the factorisation $(x^3 - 2x^2 + 1) = (x - 1)(x^2 - x - 1)$, or otherwise, show that the particle moves between two fixed limits in $r$ and find those limits.
3. A spring $OA$ of natural length $a_1$ and stiffness $k_1$ has the end $O$ fixed on a smooth horizontal surface. A particle of mass $m$ is placed on the surface and attached at $A$. A second spring $AB$ of natural length $a_2$ and stiffness $k_2$ is also attached to the mass $m_1$. A second particle of mass $m$ is placed on the surface and attached at $B$.

The system is pulled out horizontally and released so that the subsequent motion is in a straight line.

(i) (4 marks)
Let $x_1$ and $x_2$ be the positions of the masses at $A$ and $B$ at time $t$. Show that the Lagrangian for the system is

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k_1(x_1 - a_1)^2 - \frac{1}{2}k_2(x_2 - x_1 - a_2)^2.$$

(ii) (3 marks)
Find the equilibrium values of $x_1$ and $x_2$.

(iii) (2 marks)
Let $s_1$ and $s_2$ be the displacements of $A$ and $B$ from their equilibrium positions. Show that the Lagrangian is then

$$L = \frac{1}{2}m(s_1^2 + s_2^2) - \frac{1}{2}k_1s_1^2 - \frac{1}{2}k_2(s_2 - s_1)^2.$$

(iv) (2 marks)
Find the Lagrange equations of motion.

(v) (3 marks)
Writing $k_1/m = \omega_1^2$ and $k_2/m = \omega_2^2$, show that the frequencies of the normal modes of oscillation are given by

$$\omega^2 = \frac{\omega_1^2 + 2\omega_2^2 \pm \sqrt{\omega_1^4 + 4\omega_2^4}}{2}.$$
4. A sphere of mass $M$ and radius $a$ rolls, without slipping, down the face of a wedge which makes an angle $\alpha$ to the horizontal. The wedge has mass $m$ and can slide freely on a smooth horizontal table. Both sphere and wedge are at rest when the sphere is released. Let $s$ be the distance of the sphere from the apex of the wedge and $x$ be the horizontal displacement of the wedge from its initial position.

(i) (2 marks)
Assume a cartesian coordinate system with its origin at the initial position of the apex of the wedge. By drawing a diagram, or otherwise, show that the coordinates $(x_0, y_0)$ of the centre of the sphere are

\[
x_0 = x + s \cos \alpha - a \sin \alpha \\
y_0 = s \sin \alpha + a \cos \alpha
\]

(ii) (5 marks)
Show that the Lagrangian for the system is

\[
L = \frac{1}{2} (M + m) \dot{x}^2 + M \dot{x} \dot{s} \cos \alpha + \frac{7}{10} M \dot{s}^2 - Mgs \sin \alpha.
\]

(iii) (2 marks)
Write down the conjugate momentum corresponding to any ignorable coordinate.

(iv) (2 marks)
Write down the Lagrange equation for any non-ignorable coordinate.

(v) (3 marks)
Hence, or otherwise, show that

\[
\ddot{s} = \frac{g \sin \alpha}{\frac{M}{M+m} \cos^2 \alpha - \frac{7}{5}}.
\]

[The moment of inertia of a sphere of mass $M$ and radius $a$ is $\frac{2}{5}Ma^2$.]

5. (Advanced Question)
An experimental particle physicist is studying electro-weak unification and needs your help analysing an observed trajectory.
The trajectory of a particle with unit mass in plane polar coordinates $(r, \theta)$ is observed to be

\[
r = \frac{2}{\mu} \log \frac{\theta}{\theta_0}, \quad \theta > \theta_0 > 0.
\]

Let $h$ be the angular momentum per unit mass, and let $E$ be the total energy per unit mass. Assume both $h$ and $E$ are known or have been measured.

(i) (14 marks)
Assuming that the particle is moving under the action of a central force law, calculate the form of $V(r)$. 

This is the Last Page of this Extended Answer Question Paper