Tutorial Exercises: Covariance of the Lagrangian Formalism

1. (a) Consider again the four Lagrangians $L(t, x, y, \dot{x}, \dot{y})$ from a previous tutorial set.
   
   i. $\dot{x}^2 + \dot{y}^2 - k^2(x + y)^2$
   
   ii. $\dot{x}^2 + \dot{y}^2 - k^2(x - y)^2$
   
   iii. $\dot{x}^2 + \dot{y}^2 + k^2(x + y)^2$
   
   iv. $\dot{x}^2 - \dot{y}^2 + k^2(x + y)^2$

   Consider the point transformation defined by $x = u + v$ and $y = u - v$. Rewrite each of the four Lagrangians in terms of the new variables.

   (b) Notice that none of the original coordinates were ignorable. Do any of the new Lagrangians have ignorable coordinates?

   (c) Solve the equations of motion taking full advantage of any ignorable coordinates. Look back at your earlier tutorial solution of the same problems and compare the amount of work required.

1. Solution: Taking a time derivative gives $\dot{x} = \dot{u} + \dot{v}$ and $\dot{y} = \dot{u} - \dot{v}$. The four Lagrangians become

   (a) $2\dot{u}^2 + 2\dot{v}^2 - 4k^2u^2$
   
   (b) $2\dot{u}^2 + 2\dot{v}^2 - 4k^2v^2$
   
   (c) $2\dot{u}^2 + 2\dot{v}^2 + 4k^2u^2$
   
   (d) $4\dot{u}\dot{v} + 4k^2u^2$

   We look at them all at the same time to compare what happens. In the first and third examples $v$ is ignorable, and in both these cases the first integral or constant of the motion is $4\dot{v} = C_1$, which can be solved straight away to give $v = \frac{1}{4}C_1t + C_2$. In the second example $u$ is ignorable, and the first integral or constant of the motion is $4\dot{u} = C_1$, which can be solved straight away to give $u = \frac{1}{4}C_1t + C_2$. In the fourth example $v$ is ignorable, but the first integral is actually $4\dot{u} = C_1$, which can be solved straight away to give $u = \frac{1}{4}C_1t + C_2$.

   The full Euler or Lagrange equation has to be derived for the non-ignorable co-ordinate. These are in the original order above

   (a) $4\ddot{u} + 8k^2u = 0$
   
   (b) $4\ddot{v} + 8k^2v = 0$
   
   (c) $4\ddot{u} - 8k^2u = 0$
   
   (d) $4\ddot{v} + 8k^2u = 0$

   The first and second equations have the general solution $C_3 \sin \sqrt{2}t + C_4 \cos \sqrt{2}t$. The third equation has the general solution $C_3 \exp(\sqrt{2}t) + C_4 \exp(-\sqrt{2}t)$.

   The last equation simplifies when the relevant expression for $u$ is inserted and becomes $\ddot{v} + 2k^2(\frac{1}{4}C_1t + C_2) = 0$. Rearranging and integrating twice gives $v = -\frac{1}{12}C_1k^2t^3 - \frac{1}{4}C_2k^2t^2 + C_3t + C_4$.

   It’s not hard to show all these solutions are equivalent to the ones on the earlier solution sheet. Finding them here was a lot easier!
2. Consider the transformation

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = r^2. \]

This transformation enforces a constraint between \( x, y \) and \( z \). Eliminate \( r \) and \( \theta \) to find the constraint (i.e. and equation for the surface). What surface is described by this constraint?

Apply the transformation to

\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \]

2. **Solution:** To find the surface in cartesian coordinates, eliminate \( \theta \) by taking \( x^2 + y^2 \) and then eliminate \( r \). Thus,

\[ x^2 + y^2 = z \]

is the surface, which is a paraboloid.

First calculate the derivatives

\[ \dot{x} = \dot{r} \cos \theta - r \sin \theta \frac{\dot{\theta}}{\sqrt{\rho^2 + z^2}}, \quad \dot{y} = \dot{r} \sin \theta + r \cos \theta \frac{\dot{\theta}}{\sqrt{\rho^2 + z^2}}, \quad \dot{z} = 2r \dot{r}. \]

Then substitute into \( L \) and simplify

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + 4r^2 \dot{r}^2) - mg \]

3. Starting from the Kepler problem in cylindrical polar coordinates, namely,

\[ L = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + z^2) + \frac{\mu}{\sqrt{\rho^2 + z^2}}, \]

transform to parabolic coordinates defined by \( \rho = \sqrt{\xi \eta}, \phi = \phi \) and \( z = \frac{1}{2}(\xi - \eta) \).

3. **Solution:** From the transformation equations

\[ \rho^2 + z^2 = \xi \eta + \frac{1}{4}(\xi - \eta)^2 = \frac{1}{4}(\xi + \eta)^2 \]

which means

\[ \sqrt{\rho^2 + z^2} = \frac{1}{2}(\xi + \eta) \]

Also

\[ \dot{z} = \frac{1}{2}(\dot{\xi} - \dot{\eta}) \]

\[ \dot{\rho} = \frac{1}{2 \sqrt{\xi \eta}} (\xi \dot{\eta} + \eta \dot{\xi}) \]

Substituting into the Lagrangian gives

\[ L = \frac{1}{2} m \left\{ \frac{1}{4 \xi \eta} (\xi \dot{\eta} + \eta \dot{\xi})^2 + \rho^2 \dot{\phi}^2 + \frac{1}{4} (\dot{\xi} - \dot{\eta})^2 \right\} - V, \]

\[ = \frac{1}{2} m \left\{ \frac{\xi + \eta}{4 \xi \eta} (\dot{\xi}^2 + \dot{\eta}^2) + \xi \eta \dot{\phi}^2 \right\} + \frac{2 \mu}{\xi + \eta}.\]
4. Covert intelligence operative Sydney Bristow discovers an old document written by the inventive genius Milo Rimbaldi that contains the following Lagrangian

\[ L(t, e, s, \dot{e}, \dot{s}) = \frac{s^2}{s^3} + \frac{4e^2}{s(1+e^2)^2} - \frac{8ge}{\sqrt{s(1+e^2)}} \]

where \( e \) is called elevation gradient, and \( s \) is called signal strength. The only other clue as to what physical system it describes is that \( g \) stands for gravity.

(a) If you feel algebraically macho, determine the Lagrange equations directly from this Lagrangian and solve them...

OR

...find a change of variables that makes the Lagrangian simpler. Hint: If you saw \( \sqrt{1+e^2} \) in the denominator of an integral what substitution would you try? Remember \( e \) is called a gradient! Also try to get rid of the \( \sqrt{s} \) over \( s \).

(b) Keep applying more transformations if you can continue to simplify the Lagrangian.

(c) Interpret your simplified Lagrangian and give the solution to the Lagrange equations.

(d) Now go back and justify the naming of the two original co-ordinates.

4. **Solution:** The first transformation to try is \( e = \tan \theta \). Then

\[ \dot{e} = \sec^2 \theta \dot{\theta} = (1+e^2)\dot{\theta} \]

Substituting gives

\[ L(t, \theta, s, \dot{\theta}, \dot{s}) = s^{-3}s^2 + \frac{4\dot{\theta}^2}{s} - \frac{8g\tan \theta}{\sqrt{s}\sec \theta} = s^{-3}s^2 + \frac{4\dot{\theta}^2}{s} - \frac{8g}{\sqrt{s}} \sin \theta \]

Then try \( s = u^2 \) to get rid of the square-root. Thus \( \dot{s} = 2u \dot{u} \) and

\[ L(t, \theta, u, \dot{\theta}, \dot{u}) = u^{-6}4u^2u^2 + \frac{4\dot{\theta}^2}{u^2} - \frac{8g}{u} \sin \theta = 4u^{-4}u^2 + \frac{4\dot{\theta}^2}{u^2} - \frac{8g}{u} \sin \theta \]

Then try \( u = 1/r \) to simplify the terms, so \( \dot{u} = -\dot{r}/r^2 \) and

\[ L(t, \theta, r, \dot{\theta}, \dot{r}) = 4r^4r^{-4}\dot{r}^2 + 4r^2\dot{\theta}^2 - 8gr \sin \theta = 4(\dot{r}^2 + r^2\dot{\theta}^2) - 8gr \sin \theta \]

Then \( L \) looks like something in polar coordinates, so try \( x = r \cos \theta \), and \( y = r \sin \theta \) to make it look like cartesian coordinates.

\[ L(t, x, y, \dot{x}, \dot{y}) = 4(\dot{x}^2 + \dot{y}^2) - 8gy \]

It is now obvious that this Lagrangian just describes a particle of mass \( m = 8 \) falling under gravity.

The parameter \( e \) is the slope or gradient of the line-of-sight from the origin. The parameter \( s \) goes like inverse distance squared, so if we were tracking the particle with some radar or wave-based method, it would be the signal strength.

This question is meant to drive home the realisation that good choices of coordinates makes an enormous difference in both our ability to analyse the system but also to interpret the results.
5. Let \( F(x, y, z) = x^a y^b z^c \). What is the degree of \( F \). Show that
\[
x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = (a + b + c)F
\]
Note that the formula works even if \( a + b + c \) is not an integer.

Check Euler’s Homogeneous function theorem for the function
\[
F(x, y, z) = \sqrt{x^3 + y^2 + x y z}
\]
5. **Solution:** The degree of \( F \) is \( a + b + c \). Now,
\[
\frac{\partial F}{\partial x} = a x^{a-1} y^b z^c, \quad \frac{\partial F}{\partial y} = b x^a y^{b-1} z^c, \quad \frac{\partial F}{\partial z} = c x^a y^b z^{c-1}
\]
The result follows easily from there.

The quantity inside the square-root has degree 3, so the second function has degree 3/2. Now,
\[
\frac{\partial F}{\partial x} = \frac{3x^2 + y z}{2\sqrt{x^3 + y^2 + x y z}}, \quad \frac{\partial F}{\partial y} = \frac{z^2 + x z}{2\sqrt{x^3 + y^2 + x y z}}, \quad \frac{\partial F}{\partial z} = \frac{2y z + x y}{2\sqrt{x^3 + y^2 + x y z}}
\]
so
\[
x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = \frac{3x^3 + x y z + y z^2 + x y z + 2y z^2 + x y z}{2\sqrt{x^3 + y^2 + x y z}} = \frac{3}{2} F
\]

6. The Lagrangian for a particle in an electromagnetic field is
\[
L = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - eV(\mathbf{r}, t) + e \mathbf{r} \cdot \mathbf{A}(\mathbf{r}, t)
\]
Decide whether the Lagrangian can be split into homogeneous parts and the degree of each. Thus, show that the energy of a particle in an electromagnetic field does not depend on the magnetic part of the field.

6. **Solution:** The term \( \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \) is homogeneous of degree 2 in the velocities, so it should appear unchanged in the energy.

The term \( -eV(\mathbf{r}, t) \) is homogeneous of degree 0 in the velocities, so it should appear with the opposite sign in the energy.

The term \( e \mathbf{r} \cdot \mathbf{A}(\mathbf{r}, t) \) is homogeneous of degree 1 in the velocities, so it should not appear at all in the energy.

Thus
\[
E = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + eV(\mathbf{r}, t).
\]

7. Starting with
\[
\dot{q}_i = \sum_{k=1}^{m} \frac{\partial q_i}{\partial Q_k} \dot{Q}_k + \frac{\partial q_i}{\partial t}
\]
calculate \( \partial \dot{q}_i / \partial \dot{Q}_j \). Note that the old coordinates \( q_i \) do not depend on any of the new velocities \( \dot{Q}_j \).

In the new coordinate system, all the velocities \( \dot{Q} \) are treated as independent quantities. What does this imply about \( \partial \dot{Q}_k / \partial \dot{Q}_j \) when \( k = j \) and when \( k \neq j \)?

Hence show that
\[
\frac{\partial \dot{q}_i}{\partial \dot{Q}_j} = \frac{\partial q_i}{\partial \dot{Q}_j}.
\]
7. Solution:

Since the coordinates $q_i$ do not depend on the velocities $\dot{Q}_j$, the required partial derivative of $\dot{q}_i$ can be written as

$$\frac{\partial \dot{q}_i}{\partial \dot{Q}_j} = \sum_{k=1}^{m} \frac{\partial q_i}{\partial Q_k} \frac{\partial \dot{Q}_j}{\partial \dot{Q}_j}.$$

The independence of the velocities from each other in the new coordinate system means that

$$\frac{\partial \dot{Q}_k}{\partial \dot{Q}_j} = \delta_{kj}$$

Thus,

$$\frac{\partial \dot{q}_i}{\partial Q_j} = \sum_{k=1}^{m} \frac{\partial q_i}{\partial Q_k} \delta_{kj} = \frac{\partial q_i}{\partial Q_j}.$$

8. Starting with

$$\dot{q}_i = \sum_{k=1}^{m} \frac{\partial q_i}{\partial Q_k} \dot{Q}_k + \frac{\partial q_i}{\partial t}$$

calculate $\partial \dot{q}_i/\partial Q_j$. Note, in the new coordinate system, the $Q$ and the $\dot{Q}$ are treated as independent quantities.

Now using the chain rule

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{k=1}^{m} \dot{Q}_k \frac{\partial}{\partial Q_k}$$

and the definition of $J_{ij}$ calculate $\frac{d}{dt}J_{ij}$.

Show that the two results are the same. [What technical condition do you need about mixed second order partial derivatives.]

8. Solution: First

$$\frac{\partial \dot{q}_i}{\partial Q_j} = \sum_{k=1}^{m} \frac{\partial^2 q_i}{\partial Q_j \partial Q_k} \dot{Q}_k + \frac{\partial^2 q_i}{\partial Q_j \partial t}$$

Also

$$\frac{d}{dt}J_{ij} = \frac{d}{dt} \left( \frac{\partial q_i}{\partial Q_j} \right) = \sum_{k=1}^{m} \frac{\partial}{\partial Q_k} \left( \frac{\partial q_i}{\partial Q_j} \right) \dot{Q}_k + \frac{\partial}{\partial t} \left( \frac{\partial q_i}{\partial Q_j} \right)$$

the two expressions are equal if we assume that mixed second-order partial derivatives commute.