
Write down Hamilton’s equations for the following Hamiltonian

\[ H = \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) - \frac{\mu m}{r}. \]

1. **Solution:** Hamilton’s equations are

\[
\begin{align*}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2 \theta} \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_r^2}{mr^2} + \frac{p_\theta^2}{mr^2 \sin^2 \theta} - \frac{\mu m}{r^2} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \left( \frac{p_\phi^2 \cos \theta}{mr^2 \sin^2 \theta} \right) \\
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 0
\end{align*}
\]

2. A time-dependent Hamiltonian

Write down Hamilton’s equations for the Hamiltonian

\[ H(t, q, p) = \frac{p^2}{2m} - mCtq \]

where C is a constant. Solve these equations for the initial conditions \( q = 0, p = p_0 \) at \( t = 0 \).

2. **Solution:**

Hamilton’s equations are

\[
\begin{align*}
\dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\
\dot{p} &= -\frac{\partial H}{\partial q} = mCt.
\end{align*}
\]

The second equation can be integrated immediately to give

\[ p(t) = \frac{1}{2} mCt^2 + p_0 \]

where the initial condition is also incorporated. Substituting into the first equation gives

\[ \dot{q} = \frac{1}{2} Ct^2 + p_0/m \]

which integrates to

\[ q(t) = \frac{1}{6} Ct^3 + \left( \frac{p_0}{m} \right) t \]

where again the initial condition has been used.
3. Consider the following transformation mentioned briefly in the text

\[ Q = \frac{p}{2q}, \quad P = -q^2 \]

(a) Determine the inverse transformation.
(b) Find the new Hamiltonian \( H(Q, P) \) by applying the transformation to the following Hamiltonian

\[ H = \frac{1}{2}p^2 + \frac{1}{2}q^2. \]

(c) Write down Hamilton’s equations for both the original and the transformed system.
(d) Solve whichever system of Hamilton’s equations look nicer to you.
(e) Transform your solution into the other set of variables and confirm that it satisfies the other system of Hamilton’s equations.

3. Solution:

(a) The inverse transformation is obtained by first noting that \( q = \sqrt{-P} \) and then \( p = 2qQ = 2\sqrt{-PQ} \).

(b) Substituting the above into the Hamiltonian gives

\[ H(Q, P) = \frac{1}{2}4(-P)Q^2 + \frac{1}{2}(-P) = -\frac{1}{2}P(1 + 4Q^2). \]

(c) The original Hamilton’s equations of motion are

\[ \dot{q} = \frac{\partial H}{\partial p} = p \]

and

\[ \dot{p} = -\frac{\partial H}{\partial q} = -q. \]

The new Hamilton’s equations of motion are

\[ \dot{Q} = \frac{\partial H}{\partial P} = -\frac{1}{2}(1 + 4Q^2). \]

and

\[ \dot{P} = -\frac{\partial H}{\partial q} = 4PQ. \]

(d) The original equations look nicer and are coupled linear ODEs, so we know that the solutions are trig functions with frequency \( \omega = 1 \) and \( m = 1 \) and can be written in the form

\[ q(t) = q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) \]

\[ p(t) = -m\omega q_0 \sin(\omega t) + p_0 \cos(\omega t) \]

(e) The last part is just long algebra to confirm that

\[ Q(t) = \frac{p}{2q} = \frac{1}{2} \frac{-m\omega q_0 \sin(\omega t) + p_0 \cos(\omega t)}{q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t)} \]

\[ P(t) = -q^2 = -\left[q_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t)\right]^2 \]

satisfies

\[ \dot{Q} = \frac{1}{2}(1 + 4Q^2) \quad \dot{P} = 4PQ. \]
4. Show that the time-independent transformation

\[ P = p + q^2 + pq^2, \quad Q = \tan^{-1}q \]

is canonical by obtaining a suitable generating function of type \( F_3(p, Q) \).

4. **Solution:**

The inverse transformation is

\[
q = \tan Q \\
p = \frac{P - q^2}{1 + q^2} = \frac{P - \tan^2 Q}{\sec^2 Q} = P \cos^2 Q - \sin^2 Q
\]

Making \( P \) and \( q \) the subjects and moving \( q \) and \( P \) to RHS gives

\[
q = \tan Q \\
P = p(1 + q^2) + q^2 = p \sec^2 Q + \tan^2 Q
\]

We want

\[
q = \tan Q = -\frac{\partial F_3}{\partial p} \\
P = p \sec^2 Q + \tan^2 Q = -\frac{\partial F_3}{\partial Q}
\]

Thus \( F_3 = -p \tan Q - \tan Q + Q \) can be obtained by integrating both expressions and forcing consistency.

5. Find the conditions that need to be satisfied by the real constants \( \alpha, \beta, \gamma \) and \( \delta \) so that the time-independent transformation

\[ Q = \alpha pq^\gamma, \quad P = \beta q^\delta \]

is canonical.

5. **Solution:** Rearrange the transformation first to produce a mixture of old and new variables on each side. Thus

\[
p = \alpha^{-1}Qq^{-\gamma}, \\
P = \beta q^\delta
\]

Then we get the conditions

\[
p = \frac{\partial F_1}{\partial q} = \alpha^{-1}Qq^{-\gamma}, \\
P = -\frac{\partial F_1}{\partial Q} = \beta q^\delta
\]

To prove it is canonical form the mixed second derivatives and compare

\[
\frac{\partial^2 F_1}{\partial Q \partial q} = \alpha^{-1}q^{-\gamma}, \\
\frac{\partial^2 F_1}{\partial q \partial Q} = -\beta q^{\delta - 1}
\]

thus one immediately sees that \( \alpha \beta \delta = -1 \) and \( -\gamma = \delta - 1 \) as above. If one actually wants to find \( F_1 \) then integrate either condition to give \( F_1 = -\beta q^\delta Q \). Thus the transformations form a two-parameter family of transformations. [They are NOT necessarily what is called a Lie group, for those who know and are interested. Not all families of transformations, are groups of transformations.]
6. Find a canonical transformation that transforms the Hamiltonian
\[ H = \frac{1}{2}(p^2 q^4 + \frac{1}{q^2}) \]
to that of the one dimensional simple harmonic oscillator, and hence solve the dynamical problem for the original Hamiltonian. Verify that your solution satisfies Hamilton’s equations for the original Hamiltonian.

6. Solution: The idea is to use one of the canonical transformations from the previous question. Noticing that \( P \) only depends on \( q \), but \( Q \) depends on both \( p \) and \( q \), we try to make the first term in \( H \) look like \( Q^2 \) and the second term look like \( P^2 \). Thus, choose \( \delta = -1 \). This forces \( \gamma = 2 \) and \( \beta = 1/\alpha \). Substituting yields
\[ \mathcal{H}(P,Q) = \frac{1}{2}(\frac{Q^2}{\alpha^2} + \alpha^2 P^2) \]
You can choose \( \alpha \) anyway you like, the Hamiltonian will still represent some simple harmonic oscillator. \( \alpha = 1 \) makes the arithmetic nicer.

Hamilton’s eqns become
\[ \dot{Q} = P, \quad \dot{P} = -Q \]
with the obvious solutions
\[ Q(t) = Q_0 \cos(t) + P_0 \sin(t), \]
\[ P(t) = P_0 \cos(t) - Q_0 \sin(t). \]
Thus, the original problem has solutions
\[ q(t) = \frac{1}{P_0 \cos(t) - Q_0 \sin(t)}, \]
\[ p(t) = (Q_0 \cos(t) + P_0 \sin(t))(P_0 \cos(t) - Q_0 \sin(t))^2. \]

7. Electromagnetism

Starting from the Lagrangian for the electromagnetic interaction
\[ L = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - eV(\mathbf{r},t) + e \mathbf{r} \cdot \mathbf{A}(\mathbf{r},t). \]
determine the corresponding Hamiltonian.

You may work in cartesian co-ordinates if you prefer, but try to re-express your final answer in vector notation.

7. Solution: The fastest method is to simply realise that the quadratic term has a symmetric matrix which has \( m \) along the diagonal: the inverse will just have \( 1/m \). The term linear in the velocities has \( e \mathbf{A} \) as the prefactor. Thus the conjugate momentum is \( p = m \dot{\mathbf{r}} + e \mathbf{A} \). Thus, using the short-cut given in the notes the Hamiltonian is
\[ H = \frac{1}{2m}[\mathbf{p} - e \mathbf{A}] \cdot [\mathbf{p} - e \mathbf{A}] + eV(\mathbf{r},t). \]