6.0.1 Stationary conditions for an AR process

Suppose that an AR(p) process given by

\[ X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_p X_{t-p} + Z_t \]

where \( Z_t \sim WN(0, \sigma^2) \).

Let \( \alpha(B) = I - \alpha_1 B - \alpha_2 B^2 - \cdots - \alpha_p B^p \) be the corresponding AR(p) polynomial. It is known that \( \alpha(B) \) can be factorized into \( p \) linear factors of the form

\[ \alpha(B) = (I - \theta_1 B)(I - \theta_2 B)\cdots(I - \theta_p B) \]

and therefore

\[ X_t = [\alpha(B)]^{-1}Z_t = (I - \theta_1 B)^{-1}(I - \theta_2 B)^{-1}\cdots(I - \theta_p B)^{-1}Z_t. \]

Recall that each \( (I - \theta_i B)^{-1}, \ i = 1, 2, \cdots, p \) is convergent if \( |\theta_i| < 1 \) or the root of \( 1 - \theta_i \omega = 0 \) is outside the unit circle for each \( i = 1, 2, \cdots, p \).

We summarize the above in the theorem below.

**Theorem 1:** An AR(p) process is stationary iff the roots of the associated AR(p) polynomial \( \alpha(\omega) \) has zeros outside the unit circle.
The next theorem gives the stationary solution for an AR(p) process.

**Theorem 2:** An AR(p) process satisfying stationary conditions as in Theorem 1 can be written as

\[ X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \]

where \( \sum_{j=0}^{\infty} \psi_j < \infty \).

**Note:** This is known as the stationary solution.

### 6.0.2 Analysis 1st order AR process or an AR(1)

On page 49 we have considered the AR(1) process given by \( X_t = \alpha X_{t-1} + Z_t \) and have shown that it is stationary if \(|\alpha| < 1\).

The corresponding stationary solution is

\[ X_t = \sum_{j=0}^{\infty} \alpha^j Z_{t-j}, \]

### 6.0.3 Analysis 2nd order AR process or an AR(2)

The process generated by the recursive equation

\[ X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t \]

is called an AR(2) process. This can be written as:

\[ (1 - \alpha_1 B - \alpha_2 B^2) X_t = Z_t \]

or

\[ \alpha(B) X_t = Z_t, \]

where \( \alpha(B) = 1 - \alpha_1 B - \alpha_2 B^2 \) is the corresponding AR(2) operator or AR(2) polynomial.

Suppose that \( \alpha(B) \) can be factorized as

\[ \alpha(B) = (1 - \theta_1 B)(1 - \theta_2 B). \]

Since each factor is an AR(1) operator, this AR(2) process is stationary provided both \(|\theta_1| < 1\) and \(|\theta_2| < 1\) or the roots of \( \alpha(\omega) = 0 \) lie outside the unit circle.

**Reason:** Roots of \( \alpha(\omega) = 0 \) are given by \( \omega_1 = \frac{1}{\theta_1}, \omega_2 = \frac{1}{\theta_2} \). Since \(|\theta_1| < 1, |\theta_2| < 1\), we have \(|\omega_1| > 1\) and \(|\omega_2| > 1\).
Example: Consider the AR(2) process given by $X_t = 1.3X_{t-1} - 0.4X_{t-2} + Z_t$ i.e. $(1 - 1.3B + 0.4B^2)X_t = Z_t$. Determine whether $\{X_t\}$ is stationary.

Solution: Let $\alpha(B) = 1 - 1.3B + 0.4B^2$ and find the roots of $\alpha(\omega) = 0$.

Clearly, $\alpha(\omega) = (1 - 0.5\omega)(1 - 0.8\omega)$ and $\alpha(\omega) = 0$ gives $\omega = 1.25$ or $\omega = 2$.

Since both roots are outside the unit circle, the given AR(2) process is stationary.

An alternative method:

When the polynomial $\alpha(\omega)$ have no factors, use the quadratic formula to find its solution. That is the roots of $0.4\omega^2 - 1.3\omega + 1 = 0$ are given by

$$\omega = \frac{1.3 \pm \sqrt{1.3^2 - 4 \times 0.4}}{2 \times 0.4} = 1.25, 2$$

Since both roots are $> 1$ (or outside the unit circle), $\{X_t\}$ is stationary.

In R use `polyroot(c(1,-1.3,0.4))`

Example: Find the stationary solution for the above AR(2) process.

Solution:

Write $(1 - 0.5B)(1 - 0.8B)X_t = Z_t$ and hence $X_t = (1 - 0.5B)^{-1}(1 - 0.8B)^{-1}Z_t$.

Now we have $X_t = (\sum_{j=0}^{\infty} 0.5^j B^j)(\sum_{j=0}^{\infty} 0.8^j B^j)Z_t$ and this can be written as $X_t = \sum_{j=0}^{\infty} \delta_j Z_{t-j}$, where $\delta_0 = 1$, $\delta_1 = 1.3$, $\delta_2 = 0.5^2 + 0.5 \times 0.8 = 0.8^2$ and $\delta_j = 0.5^j + 0.5^{j-1} \times 0.8 + 0.5^{j-2} \times 0.8^2 + \cdots + 0.5 \times 0.8^{j-1} + 0.8^j$, $j \geq 1$.

Note: The above solution is known as the stationary solution to the given AR(2) process. In general any stationary AR(2) process is equivalent to an MA($\infty$) given by

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

where $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

A quick check for stationarity of an AR(2) process

Since the AR(2)m polynomial is quadratic, using the ther of quadratic equations we have the following: If $X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t$ is stationary, then it can be seen that $|\alpha_2| < 1$, $\alpha_2 + \alpha_1 < 1$, and $\alpha_2 - \alpha_1 < 1$.

In the above example, $\alpha_1 = 1.3$ and $\alpha_2 = -0.4$. It is easy to verify that $|\alpha_2| < 1$, $\alpha_2 + \alpha_1 < 1$, and $\alpha_2 - \alpha_1 < 1$.

This can be used as a quick check for stationarity of any AR(2) process.
**Example:** For the AR(2) process \( X_t = -0.3X_{t-1} + 0.1X_{t-2} + Z_t \) check whether all three inequalities for the parameters are satisfied.

Solution:

\[
\begin{align*}
\alpha_1 &= , \quad \alpha_2 = \\
|\alpha_2| &= \\
\alpha_2 + \alpha_1 &= \\
\alpha_2 - \alpha_1 &= 
\end{align*}
\]

and hence this AR(2) process is ....................

**Exercise:** Find the zeros of the polynomial \( \alpha(\omega) = 1 + 0.3\omega - 0.1\omega^2 \) and check whether both are outside the unit circle.

**Exercise:** Consider the AR(2) process \( X_t = 1.5X_{t-1} - 0.4X_{t-1} + Z_t \). Determine whether this process is stationary.

Solution:

\[
\begin{align*}
\alpha_1 &= , \quad \alpha_2 = \\
|\alpha_2| &= \\
\alpha_2 + \alpha_1 &= \\
\alpha_2 - \alpha_1 &= 
\end{align*}
\]

Therefore \( \{X_t\} \) is ..............................

**Exercise:** Show that the process given by \( X_t = X_{t-1} + 0.16X_{t-2} + Z_t \) is stationary. Find the stationary solution of this AR(2) process.
Yule-Walker equations for stationary AR processes

Suppose that the AR(p) process given by

\[ X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_p X_{t-p} + Z_t \]

is stationary.

Recall that \( \text{cov}(X_t, X_{t-k}) = \gamma_k, \ E(X_{t-k}Z_t) = 0, \ k \geq 1 \) and \( E(X_tZ_t) = \sigma^2 \).

Multiplying the above equation by \( X_{t-k} \) and taking expectations yield

\[ E(X_t X_{t-k}) = \alpha_1 E(X_{t-1} X_{t-k}) + \alpha_2 E(X_{t-2} X_{t-k}) \cdots + \alpha_p E(X_{t-p} X_{t-k}) + E(Z_t X_{t-k}). \]

For \( k \geq 1 \), \( E(Z_t X_{t-k}) = 0 \) and hence we have the recursion for \( \gamma_k \):

\[ \gamma_k = \alpha_1 \gamma_{k-1} + \alpha_2 \gamma_{k-2} + \cdots + \alpha_p \gamma_{k-p} \quad (***) \]

Substituting for \( k = 1, 2, \cdots \) and noting that \( \gamma_{-j} = \gamma_j \) due to stationarity, we can write a set of equations for consecutive values of acf as follows:

\[ \gamma_1 = \alpha_1 \gamma_0 + \alpha_2 \gamma_1 + \cdots + \alpha_p \gamma_{1-p} \]
\[ \gamma_2 = \alpha_1 \gamma_1 + \alpha_2 \gamma_0 + \cdots + \alpha_p \gamma_{2-p} \]
\[ \gamma_3 = \alpha_1 \gamma_2 + \alpha_1 \gamma_0 + \cdots + \alpha_p \gamma_{3-p} \]
\[ \vdots \]
\[ \gamma_p = \alpha_1 \gamma_{p-1} + \alpha_1 \gamma_{p-2} + \cdots + \alpha_p \gamma_0 \]
\[ \gamma_{p+1} = \alpha_1 \gamma_p + \alpha_1 \gamma_{p-1} + \cdots + \alpha_p \gamma_1 \]
\[ \vdots \]

Equivalently, dividing (***') by \( \gamma_0 \) we have

\[ \rho_k = \alpha_1 \rho_{k-1} + \alpha_2 \rho_{k-2} + \cdots + \alpha_p \rho_{k-p} \quad (***) \]

**Note:** This system of equations in (**) or (***) for \( k \geq 1 \) is known as Yule-Walker equations for an AR(p) process.

**Remark:** For \( k = 0 \), \( E(Z_t X_t) = \sigma^2 \) and hence from (*) we have

\[ \gamma_0 = \alpha_1 \gamma_1 + \alpha_2 \gamma_2 + \cdots + \alpha_p \gamma_p + \sigma^2. \]
Y-W equations for an AR(1) Process

It is easy to see that the acf of an AR(1) satisfies

\[ \rho_k = \alpha \rho_{k-1}, \ k \geq 1. \]

This Yule-Walker equations for the AR(1) process can be used to calculate the ACF of an AR(1) process using \( \rho(0) = 1 \) or \( \gamma(0) = \frac{\sigma^2}{1-\alpha^2} \). We have shown that (see P49) the general expression for \( \rho_k \) is \( \rho(k) = |\alpha|^k \); \( k = 0, \pm 1, \pm 2, \ldots \) and we say the acf of a stationary AR(1) process has an exponential function. This property can be used to draw a correlogram

1. Write down the first five acf values for the above AR(1) process.

2. Sketch the correlogram for the above AR(1) process.

Y-W equations for an AR(2) Process

Suppose that the AR(2) process given by \( X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t \) is stationary. Write down the Y-W equations.

Solution:
Example: Let $\rho_k$ be the acf for the stationary AR(2) process

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t.$$ 

Find $\rho_1$, $\rho_2$ and $\rho_3$ in terms of $\alpha_1$ and $\alpha_2$.

Solution:

Example: Let $\rho_k = \alpha_1 \rho_{k-1} + \alpha_2 \rho_{k-2}$, $k \geq 1$ be written as

$$(I - \alpha_1 B - \alpha_2 B^2)\rho_k = (I - \theta_1 B)(I - \theta_2 B)\rho_k = 0.$$ 

When $\theta_1$ and $\theta_2$ are real, an expression for $\rho_k$ can be written as

$$\rho_k = A_1 \theta_1^k + A_2 \theta_2^k, \text{ for all } k \geq 0.$$ 

Using $\rho_0 = 1$ and $\rho_1 = \frac{\alpha_1}{1-\alpha_2}$, show that

$$\rho_k = \frac{(1 - \theta_2^2)\theta_1^{k+1} - (1 - \theta_1^2)\theta_2^{k+1}}{(\theta_1 - \theta_2)(1 + \theta_1 \theta_2)}, \text{ for all } k \geq 0.$$ 

Solution:
**Example:** Since $\theta_1 < 1$ and $\theta_2 < 1$, there is an exponential decay of the acf. Sketch this acf.

**Example:** When $\theta_1$ and $\theta_2$ are complexes, it can be shown that an expression for $\rho_k$ is

$$
\rho_k = \frac{(-\alpha_2)^{k/2} \sin (k\theta + \Psi)}{\sin \Psi},
$$

where $\cos \theta = \frac{\alpha_1}{2\sqrt{-\alpha_2}}$ and $\tan \Psi = \frac{(1-\alpha_2)\tan \theta}{1+\alpha_2}$.

**Solution:** Exercise. Proof not expected in the exam.

**Note:** From the above expression, it is clear that the acf has a damped harmonic behaviour with period $2\pi/\theta$ and decaying amplitude since $|\alpha_2| < 1$. In this case, the acf is said to have a quasi periodic behaviour with period $2\pi/\theta$.

**Sketch of the acf:**