1. Suppose that a stationary ARMA\((1,1)\) process is given by
\[ X_t = C + \alpha X_{t-1} + Z_t + \beta Z_{t-1}, \]
where \(C\) is a constant and \(Z_t \sim WN(0, \sigma^2)\).

(a) **State** the condition/s for stationarity of \(\{X_t\}\).
(b) **Find** its mean, \(\mu = E(X_t)\) in terms of \(C, \alpha\) and \(\beta\).
(c) Following the approach of Q3 of Tutorial week 11, show that
   i. \(\gamma_0 = \alpha \gamma_1 + \sigma^2 + \beta (\alpha + \beta) \sigma^2\),
   ii. \(\gamma_1 = \alpha \gamma_0 + \beta \sigma^2\),
   iii. \(\gamma_k = \alpha \gamma_{k-1}, k \geq 2\), where \(\gamma_k = E[(X_t - \mu)(X_{t-k} - \mu)]\) is the autovariance function at lag \(k\).
(d) **Find** \(\rho_1, \rho_2, \rho_3\) and \(\rho_4\) using the results in (c).
(e) **Find** the general form of \(\rho_k, k \geq 2\).

2. Let \(X_t = 4 + 0.25X_{t-3} + Z_t - 0.3Z_{t-2}\) be an ARMA\((p,q)\) process, where \(Z_t \sim WN(0, \sigma^2)\).

(a) **What** are the values of \(p\) and \(q\)?
(b) **Show** that \(\{X_t\}\) is stationary and find its mean, \(\mu = E(X_t)\).
(c) **Find** its stationary solution in the form of \(X_t - \mu = \sum_{j=0}^{\infty} \psi_j Z_{t-j}\). Find \(\psi_j, j \geq 0\).
(d) **Find** the conditional mean, \(E(X_t|F_{t-k})\) for all \(k \geq 1\), where \(F_{t-k}\) denotes the available historical information set given by 
\(F_{t-k} = \{X_{t-k}, X_{t-k-1}, \ldots, Z_{t-k}, Z_{t-k-1}, \ldots\}\).

3. The following sample autocorrelations were calculated from four time series (i) to (iv) of 100 observations each.

<table>
<thead>
<tr>
<th>Lag</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>(r_k)</td>
<td>0.34</td>
<td>0.02</td>
<td>-0.10</td>
<td>0.15</td>
</tr>
<tr>
<td>(ii)</td>
<td>(r_k)</td>
<td>0.93</td>
<td>-0.01</td>
<td>0</td>
<td>0.05</td>
</tr>
<tr>
<td>(iii)</td>
<td>(r_k)</td>
<td>-0.26</td>
<td>-0.10</td>
<td>0.16</td>
<td>0</td>
</tr>
<tr>
<td>(iv)</td>
<td>(r_k)</td>
<td>-0.80</td>
<td>0.65</td>
<td>-0.51</td>
<td>0.30</td>
</tr>
</tbody>
</table>

(a) Determine which time series can be modelled by an MA\((q)\) process for some \(q\).
(b) If an MA\((1)\) model is appropriate, justify your suggestions using a suitable test.
(c) Estimate the parameter \(\beta\) in (b) using the method of moments.
(d) Which additional information do you require in (a) and (b) to make a good decision?

4. 100 observations on a regular time series is analysed and gave the following results for the ACF and PACF:

<table>
<thead>
<tr>
<th>Lag</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACF</td>
<td>0.806</td>
<td>0.428</td>
<td>0.078</td>
<td>-0.169</td>
<td>-0.266</td>
<td>-0.212</td>
<td>-0.044</td>
<td>0.164</td>
</tr>
<tr>
<td>PACF</td>
<td>0.806</td>
<td>-0.634</td>
<td>0.080</td>
<td>-0.061</td>
<td>0.001</td>
<td>0.170</td>
<td>0.107</td>
<td>0.112</td>
</tr>
</tbody>
</table>

Justify the choice of an AR\((2)\) for this situation by considering the ACF and PACF values individually and jointly.

PTO for Week12 Computer Exercise
Computer Exercise - Week 12

Suppose that the vector $d$ contains a set of stationary time series data. Assume that the analysis suggests an ARMA($p,q$) for the data. The maximum likelihood method can be used to estimate its parameters.

Command:

\[
\text{arima}(d, \text{order} = c(p,0,q))
\]

Problems

Simulate 300 values from the AR(2) or ARMA(2,0) model

\[
X_t = 0.6X_{t-1} - 0.4X_{t-2} + Z_t, \quad Z_t \sim NID(0,1)
\]

and save the last 200 values in $d$ to be used in Q1 to Q4 below:

1. Plot the time series, acf and pacf for this data. Are the acf and pacf patterns consistent with the theory of stationary AR(2) or ARMA(2,0)?

2. Use the data in $d$ and \texttt{arima} command to obtain the maximum likelihood estimates for the parameters in the model,

\[
X_t = \alpha_1X_{t-1} + \alpha_2X_{t-2} + Z_t.
\]

Report your estimates of $\alpha_1$ and $\alpha_2$ and their corresponding standard errors.

3. Use the first 100 values in your series $d$ to fit the AR(2) model in Q2. Note how the standard errors of your estimates change. Is this surprising?

4. Use the \texttt{arima} command to obtain the estimates for the parameters $\alpha$ and $\beta$ in the following incorrect ARMA(1,1) model for the data in $d$:

\[
X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}.
\]

Report your estimates of $\alpha$ and $\beta$ and their standard errors. Compare the estimates of $\sigma^2$ and AIC values for the correct AR(2) model in Q2 and this incorrect ARMA(1,1) model. What do you notice?

5. For the data set \texttt{beer} find the \texttt{ts.plot} and obtain a stationary series by differencing. Use the acf and pacf to suggest an appropriate ARMA model. Find the mean of your differenced data set. How does this relate to the trend in the series?