THE UNIVERSITY OF SYDNEY

Assignment 1	STAT3011/3911 Stochastic Processes	2009.

Due by 5 pm, Friday May 15, Carslaw 8th Floor, assignment box STAT3XXX.

1. A finite Markov chain with states $S = \{1, 2, ..., 8\}$ has transition matrix $P = \{p_{ij}\}, i, j = 1, 2, ..., 8$ where the following entries are positive

$$p_{16}, p_{24}, p_{25}, p_{31}, p_{38}, p_{48}, p_{52}, p_{55}, p_{63}, p_{64}, p_{75}, p_{77}, p_{78}, p_{86}$$

   All other transition probabilities are zero.

   (a) Write out the transition matrix (put "+" for a positive entry).

   (b) Find all recurrent and transient classes of states.

   (c) Find the period of each class, giving reasons.

   (d) Put the transition matrix in canonical form.

   (e) If $p_{24} = p_{52} = p_{75} = \frac{1}{2}, p_{78} = \frac{1}{4}$, write down the matrix Q and hence find the mean number of transitions to absorption from each of the transient states. (Remember that all row sums of P are 1.)

   (f) If $p_{31} = p_{63} = \frac{1}{2}$, find the stationary distribution vector corresponding to each class of recurrent states, given $\pi_6 = \frac{1}{3}$.

2. Let $\{X_n\}, n = 0, 1, 2, \cdots$ be a Markov Chain on the state space $\{1, 2, 3\}$ with transition matrix P whose only positive entries are $p_{11} = p_{12} = p_{13} = 1/3$; p_{23}; and p_{31}.

   (a) Calculate P^2 and prove all entries of P^n are positive for $n = 3, 4, \cdots$.

   (b) Find the mean recurrence time of each state.

   (c) Let $A = \{2, 3\}$. Assume that the chain started with its stationary distribution. Show that

$$P(X_{n+1} = 1 | X_n \in A) = 2/3 \text{ but that } P(X_{n+1} = 1 | X_n \in A, X_{n-1} \in A) = 1.$$

   Does this contradict the Markov property of the Markov chain? Explain (in two lines).

PLEASE TURN OVER

3. A Markov chain on states $S = \{0, 1, 2, \cdots, N-1, N\}$ has transition probabilities for $1 \le i \le N-1$ given by $p_{i,i-1} = q, p_{ii} = r, p_{i,i+1} = p$, where $q > 0, r \ge 0, p > 0, q+r+p = 1$. Denote by m_i the mean number of transitions to reach one or other of the states $\{0, N\}$, starting from $i, 1 \le i \le N-1$.

   (a) Show that
   $$m_i = 1 + qm_{i-1} + rm_i + pm_{i+1}, \quad i = 1, \cdots, N-1,$$
   where $m_0 = m_N = 0$.

   (b) Suppose now that $p = q$. Show that a specific solution to (a), without the boundary conditions is of form $m_i = Ki(N-i), i = 0, 1, \cdots, N-1, N$, where K is a constant, and find K.

   (c) Continue to suppose that $p = q$. Find the solution to (a) with boundary conditions. (Check your solution in the case $p = 1/2$ against Theorem 1.10.)

   (d) What is the mean number of visits to state i, starting from $i, i = 1, 2, \cdots N-1$ before exiting from i when $p = q$? Hence explain your answer to (c) (2 lines of writing), by relating to the case $p = 1/2$.

4. **STAT3911 only.** Suppose P is the transition matrix of an irreducible (possibly periodic, possibly infinite state - space) Markov chain $\{X_n\}$. Suppose that there exists a vector $\mathbf{x} = \{x_i\} \ge \mathbf{0}$ satisfying
   $$\mathbf{x}^{\mathbf{T}} P \le \mathbf{x}^{\mathbf{T}}, \qquad 0 < \mathbf{x}^{\mathbf{T}}\mathbf{1} < \infty.$$

   (a) Show that $\mathbf{x} > \mathbf{0}$. (That is: all elements of the vector $\mathbf{x}$ are strictly positive.) **Hint:** Is $x_i p_{ij}^{(n)} \le x_j$?

   (b) Show that the Markov chain $\{X_n\}$ has a stationary distribution. **Hint:** Suppose there is strict inequality in at least one position in the system
   $$\mathbf{x}^{\mathbf{T}} P \le \mathbf{x}^{\mathbf{T}}$$
   and obtain a contradiction.

   (c) Show that the Markov chain $\{\bar{X}_n\}$ whose transition matrix is $\bar{P} = \frac{1}{2}(I + P)$, and $\{X_n\}$ have the same set of stationary distributions. (That is, any stationary distribution of one is a stationary distribution of the other.)

   (d) Use Theorem A.2 to show that $\{\bar{X}_n\}$ is positive-recurrent, and hence show that $\{X_n\}$ has a *unique* stationary distribution all of whose elements are strictly positive.

END