THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 1

SS1001: Differential Calculus — Summer School, 2012

Web Page: `http://www.maths.usyd.edu.au/u/UG/SS/SS1001/`

Preparatory questions

1. Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$. Find

(a) $A \cup B$

Solution: The union of the sets A and B equals $\{1,2,3,4,5,6\}$.

(b) $A \cap B$

Solution: The intersection of the sets A and B equals $\{3,4\}$.

(c) $A \setminus B$

Solution: The set consisting of the elements of A that are not in B is $\{1,2\}$.

(d) $B \setminus A$

Solution: The set consisting of the elements of B that are not in A is $\{5,6\}$.

(e) $(A \setminus B) \cup (B \setminus A)$.

Solution: The union of the sets $A \setminus B = \{1,2\}$ and $B \setminus A = \{5,6\}$ equals $\{1,2,5,6\}$.

2. Let $X = \left\{n \in \mathbb{Z} \mid n^2 \leq 5\right\}$. Rewrite X as a list of numbers.

Solution: The set of integers between $-\sqrt{5}$ and $\sqrt{5}$ (inclusive) is $X = \{-2,-1,0,1,2\}$.

Are the following statements true or false?

(a) $X \subseteq \mathbb{Z}$

Solution: True. All the elements of X are integers.

(b) $X \supseteq \mathbb{Z}$

Solution: False. Not all integers are elements of X.

(c) $5 \in X$

Solution: False. 5 exceeds $\sqrt{5}$, and so is outside X.

(d) $-2 \notin X$

Solution: False. -2 is indeed an element of X.

3. If $z = 2 - i$ and $w = -4 + 3i$, find

(a) $z + w$

Solution: $z + w = (2-i) + (-4+3i) = (2-4) + (-1+3)i = -2+2i$. This can also be written $-2(1-i)$.

(b) $z - w$

Solution: $z - w = (2-i) - (-4+3i) = (2+4) + (-1-3)i = 6-4i$. This can also be written $2(3-2i)$.

(c) $|z|$

Solution: $|2-i| = \sqrt{2^2+(-1)^2} = \sqrt{2^2+1^2} = \sqrt{5}$. The first step can be omitted.

(d) $\overline{w}$

Solution: The complex conjugate of w equals $\overline{w} = -4-3i$. This can also be written $-(4+3i)$.

(e) zw

Solution: $zw = (2-i)(-4+3i) = -8+6i+4i-3i^2 = -5+10i$. This can also be written $-5(1-2i)$.

(f) $\dfrac{z}{w}$

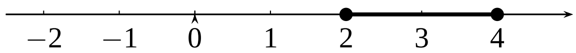
Solution: $\dfrac{z}{w} = \dfrac{2-i}{-4+3i} = \dfrac{(2-i)(-4-3i)}{(-4+3i)(-4-3i)} = \dfrac{-8-6i+4i+3i^2}{4^2+3^2} = \dfrac{-11-2i}{25}$.
This can also be written either as $-(11+2i)/25$ or $-\frac{11}{25}-\frac{2}{25}i$.

Questions to do in class

4. Locate each of the following sets on the real number line and then express each as an interval or as a union of intervals:

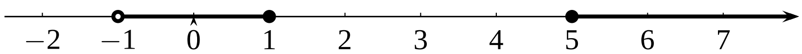
(a) $\{x \in \mathbb{R} \mid 2 \le x \le 4\}$

Solution: This is the closed interval $[2,4]$.



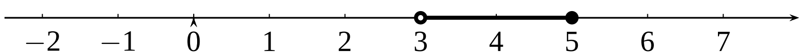
(b) $\{x \in \mathbb{R} \mid -1 < x \le 1 \text{ or } x \ge 5\}$

Solution: This is the union of two disjoint intervals, $(-1,1] \cup [5,\infty)$.



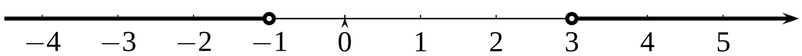
(c) $[2,5] \cap (3,6]$

Solution: This intersection of intervals can be written as a single interval, $(3,5]$.



(d) $\{x \in \mathbb{R} \mid |x-1| > 2\}$

Solution: Since $|x-1|$ can be interpreted as the distance on the number line between x and 1, the set is made up of all numbers which are more than two units from 1 on the number line. Hence, $\{x \in \mathbb{R} \mid |x-1| > 2\} = (-\infty,-1) \cup (3,\infty)$.



5. Express the following complex numbers in cartesian form:

(a) $(2+3i)+(5-6i)$

Solution: $7-3i$.

(b) $(2+3i)-(5-6i)$

Solution: $-3+9i$ or, equivalently, $-3(1-3i)$

(c) $(1+i)(1-i)$

Solution: $(1+i)(1-i) = 1-i+i-i^2 = 1+1 = 2.$

(d) $\dfrac{1+i}{1-i}$

Solution: $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = \dfrac{1+2i+i^2}{2} = i.$

(e) $(2+3i)(5-6i)$

Solution: $(2+3i)(5-6i) = 10-12i+15i-18i^2 = 28+3i.$

(f) $\dfrac{2+3i}{5-6i}$

Solution: $\dfrac{2+3i}{5-6i} = \dfrac{(2+3i)(5+6i)}{(5-6i)(5+6i)} = \dfrac{10+12i+15i+18i^2}{5^2+6^2} = \dfrac{-8+27i}{61}.$

(g) $\dfrac{1}{i} - \dfrac{3i}{1-i}$

Solution: $\dfrac{1}{i} - \dfrac{3i}{1-i} = -i - \dfrac{3i(1+i)}{1^2+1^2} = -i - \dfrac{-3+3i}{2} = \dfrac{3-5i}{2}.$

(h) $i^{123} - 4i^9 - 4i$

Solution: First you need to notice the pattern $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $\ldots$. In fact, $i^{4n} = 1$ for all $n \in \mathbb{N}$. So $i^9 = i^8 i = i$ since $i^8 = 1$ and $i^{123} = i^{120} i^3 = i^3 = -i$. Hence, $i^{123} - 4i^9 - 4i = -i - 4i - 4i = -9i$.

6. Solve the following equations over $\mathbb{C}$:

(a) $z^2 + 3z + 2 = 0$

Solution: $z^2 + 3z + 2 = (z+1)(z+2) = 0$. Hence $z = -1$ and $z = -2$.

(b) $z^2 + z + 1 = 0$

Solution: Using the formula to solve a quadratic equation, we get

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

(c) $3z^2 - 4z + 4 = 0$

Solution: Using the formula to solve a quadratic equation, we get

$$z = \frac{4 \pm \sqrt{16-48}}{6} = \frac{4 \pm \sqrt{-32}}{6} = \frac{4 \pm 4i\sqrt{2}}{6} = \frac{2 \pm 2i\sqrt{2}}{3}.$$

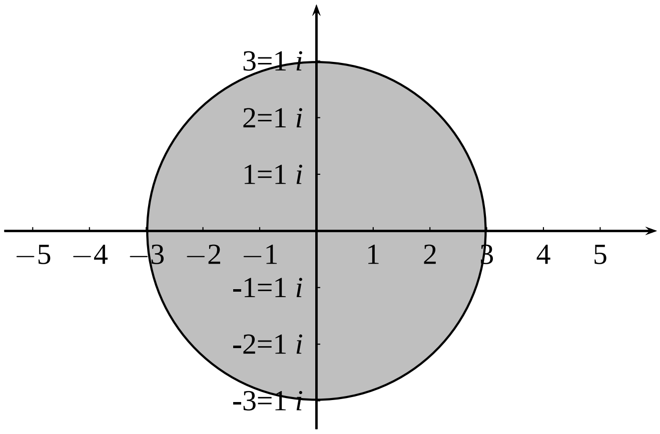
(d) $z^4 = 16$

Solution: If $z^4 = 16$, then $z^2 = 4$ or -4. Hence, $z = 2$, -2, $2i$ and $-2i$.

7. Sketch the following regions in the complex plane:

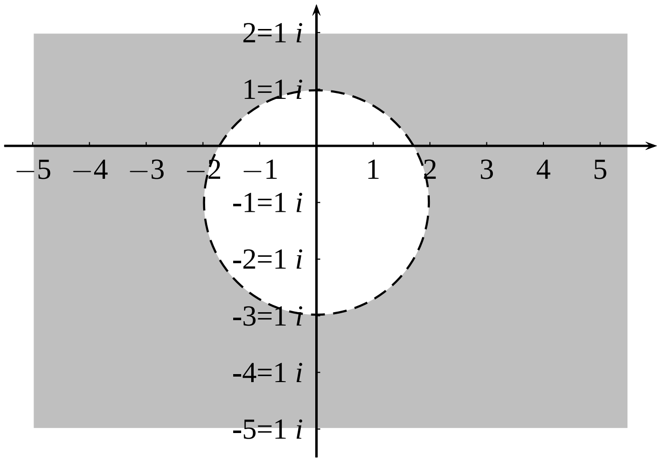
(a) $\{z \in \mathbb{C} \mid |z| \leq 3\}$

Solution: The set is shaded:



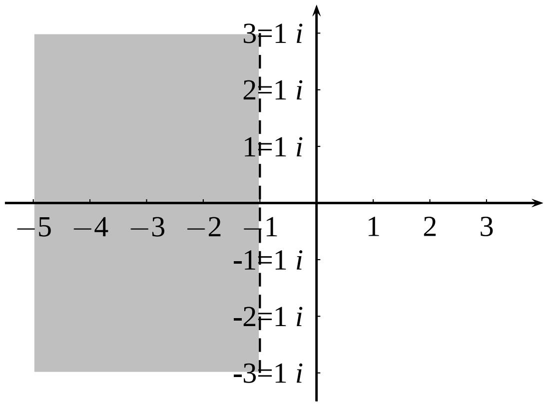
(b) $\{z \in \mathbb{C} \mid |z+i| > 2\}$

Solution: The set is shaded and does not include the broken line:



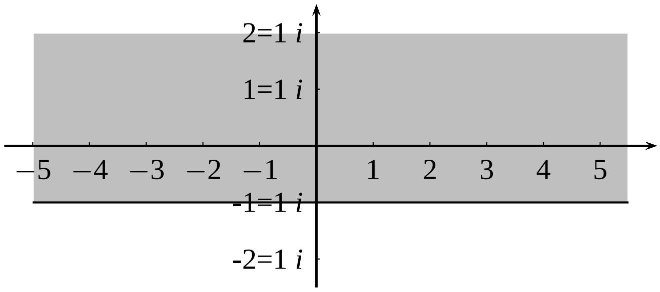
(c) $\{z \in \mathbb{C} \mid \mathrm{Re}\, z < -1\}$

Solution: The set is shaded and does not include the broken line:



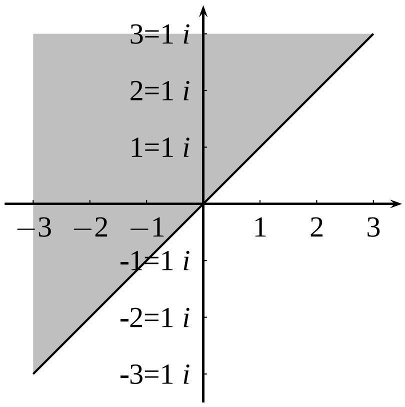
(d) $\{z \in \mathbb{C} \mid \mathrm{Im}\, z \geq -1\}$

Solution: The set is shaded:



(e) $\{z \in \mathbb{C} \mid |z-i| \leq |z-1|\}$

Solution: The set is shaded:



In this example, the required set consists of all points in the complex plane that are closer to $z = i$ than to $z = 1$ or are at the same distance. Thus, if $z = x + iy$, the solution of the inequality is the diagonal half-plane $y \geq x$. Alternatively, the inequality can be solved algebraically instead of graphically as follows:

$$z = x + iy, \qquad z - i = x + i(y-1), \qquad z - 1 = (x-1) + iy,$$
$$|z-i| = \sqrt{x^2 + (y-1)^2}, \qquad |z-1| = \sqrt{(x-1)^2 + y^2}.$$

The required inequality becomes

$$\sqrt{x^2 + (y-1)^2} \leq \sqrt{(x-1)^2 + y^2} \qquad \text{(both sides nonnegative)},$$
$$x^2 + (y-1)^2 \leq (x-1)^2 + y^2 \qquad \text{(squaring both sides)},$$
$$x^2 + y^2 - 2y + 1 \leq x^2 - 2x + 1 + y^2 \qquad \text{(expanding both sides)},$$
$$-2y \leq -2x \qquad \text{(cancelling terms)},$$
$$y \geq x \qquad \text{(reverse the inequality when signs change)}.$$

This result agrees with the graphical method.

8. (Suitable for group discussion.) Are each of these statements true or false? Explain your answer.

(a) The square of an imaginary number is always real.

Solution: True. The square of an imaginary number is always real and negative. For example, $(-7i)^2 = -49$, $(i/2)^2 = -1/4$. Every imaginary number has the form yi for some $y \in \mathbb{R}$. Squaring gives $(yi)^2 = -y^2 \in \mathbb{R}$.

(b) It does not make sense to write $|z| > |w|$ when z and w are complex numbers because the complex numbers are not ordered.

Solution: False. The complex numbers are not ordered, so for $z, w \in \mathbb{C}$, we cannot write statements such as $z > w$. However the *modulus* of a complex number is a *real* number. That is $|z|$ and $|w|$ are real and so it does make sense to write statements such as $|z| > |w|$.

(c) Real numbers cannot be graphed on the complex plane.

Solution: False. All real numbers lie on the real axis of the complex plane.

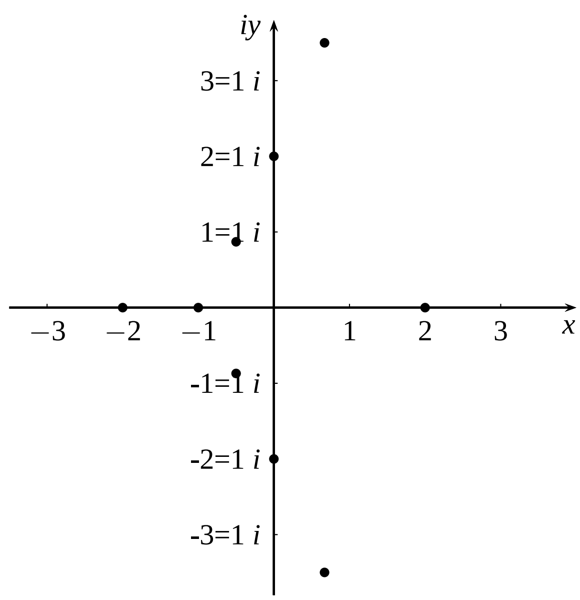
(d) When a real number is divided by a complex number the answer can never be real.

Solution: False. When a real number is divided by a complex number the answer can be real or non-real, depending on the numbers themselves. For example, $3/(1+2i)$ is non-real but $0/(1+2i)$ is real and $5/7$ is real (here we are regarding 7 as the complex number $7+0i$).

Questions for further practice

9. On the complex plane, graph the solutions to the equations in Question 6. What pattern do you notice in each pair of solutions? Explain why this pattern occurs.

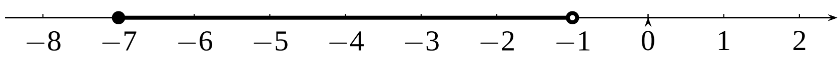
Solution: Below are the solutions to the equations in Question 6 plotted on the complex plane. As you plot them, you should notice that when there are complex solutions, these occur in pairs that are the image of one another when reflected in the real axis. In the case of quadratics, it is easy to see why this happens. Taking square roots in the quadratic formula gives terms of the form $a \pm bi$. Hence both solutions have the same real part but have imaginary parts of the opposite sign but the same magnitude.



10. Locate the following sets, which are given in interval notation, on the real number line. Rewrite each set using $\{\ldots \mid \ldots\}$ notation.

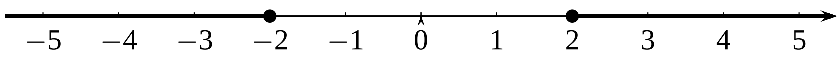
(a) $[-7,-1)$

Solution: $[-7,-1) = \{x \in \mathbb{R} \mid -7 \leq x < -1\}$.



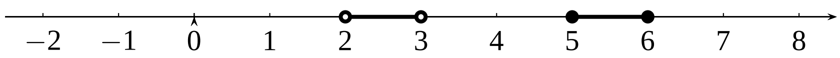
(b) $(-\infty,-2] \cup [2,\infty)$

Solution: $(-\infty,-2] \cup [2,\infty) = \{x \in \mathbb{R} \mid x \leq -2 \text{ or } x \geq 2\} = \{x \in \mathbb{R} \mid x^2 \geq 4\}$.



(c) $(2,3) \cup [5,6]$

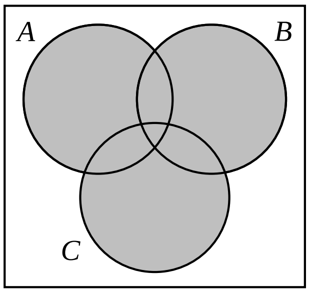
Solution: $(2,3) \cup [5,6] = \{x \in \mathbb{R} \mid 2 < x < 3 \text{ or } 5 \leq x \leq 6\}$.



11. Use a Venn diagram with three sets A, B and C to show the following:

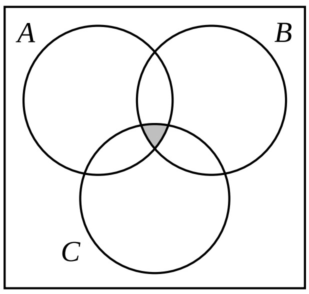
(a) $A \cup B \cup C$

Solution:



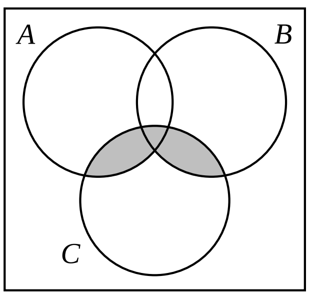
(b) $A \cap B \cap C$

Solution:



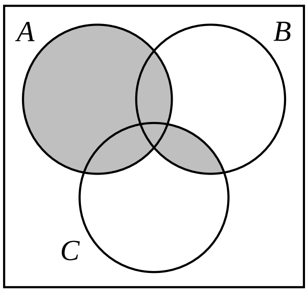
(c) $(A \cup B) \cap C$

Solution:



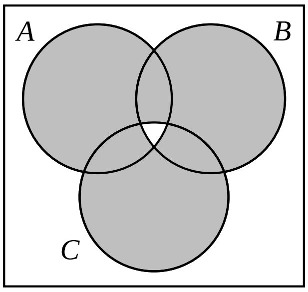
(d) $A \cup (B \cap C)$

Solution:



(e) $(A \cup B \cup C) \setminus (A \cap B \cap C)$

Solution:



(f) $\big(A \setminus (B \cup C)\big) \cup \big(B \setminus (A \cup C)\big) \cup \big(C \setminus (A \cup B)\big)$

Solution: