THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial 10

SS1001: Differential Calculus — Summer School, 2012

Web Page: `http://www.maths.usyd.edu.au/u/UG/SS/SS1001/`

Preparatory questions

1. Using the definitions of $\cosh x$ and $\sinh x$,

(a) find $\cosh(0)$ and $\sinh(0)$;

Solution: The hyperbolic functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Hence, for the case $x = 0$,

$$\cosh 0 = \frac{1+1}{2} = 1, \qquad \sinh 0 = \frac{1-1}{2} = 0.$$

(b) show that $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$.

Solution: Replace x by $-x$ and get

$$\begin{aligned}\cosh(-x) &= \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x;\\ \sinh(-x) &= \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -\sinh x.\end{aligned}$$

So $\cosh x$ (like $\cos x$) is an even function and $\sinh x$ (like $\sin x$) is an odd function.

2. Apply the horizontal line test and also the definition of injectivity to determine whether the functions given by the following formulas are injective (one-to-one) or not. Assume that all of them have domain $\mathbb{R}$.

(a) $f(x) = x + 1$

Solution: The function f is injective because if $f(x_1) = f(x_2)$ then $x_1 + 1 = x_2 + 1$, so $x_1 = x_2$. The graph of f passes the horizontal line test.

(b) $g(x) = e^x$

Solution: The function g is injective because if $g(x_1) = g(x_2)$ then $e^{x_1} = e^{x_2}$, which implies $x_1 = x_2$. The graph of g passes the horizontal line test.

(c) $f(x) = x^2 - 3$

Solution: The function f is not injective because $f(-1) = f(1) = -2$, so we can have $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$. The graph of f fails the horizontal line test.

(d) $g(x) = \sin x$

Solution: The function g is not injective because $g(0) = g(\pi) = 0$, so we can have $g(x_1) = g(x_2)$ for some $x_1 \neq x_2$. The graph of g fails the horizontal line test.

Questions to do in class

3. Find a formula for $f^{-1}(x)$ when $f(x)$ is given by the following formulas. (You may assume that the functions are injective on their natural domains.) In each case write down the domain and range of $f^{-1}(x)$.

(a) $f(x) = e^{x+1}$

Solution: The domain of f is $\mathbb{R}$ and the range is $(0,\infty)$. Put $y = e^{x+1}$. Then $\ln y = x+1$ so $x = \ln y - 1 = f^{-1}(y)$. Interchanging x and y gives

$$f^{-1}(x) = \ln x - 1.$$

The domain of f^{-1} is $(0,\infty)$ and range of f^{-1} is $\mathbb{R}$.

(b) $f(x) = 3 - \sinh x$

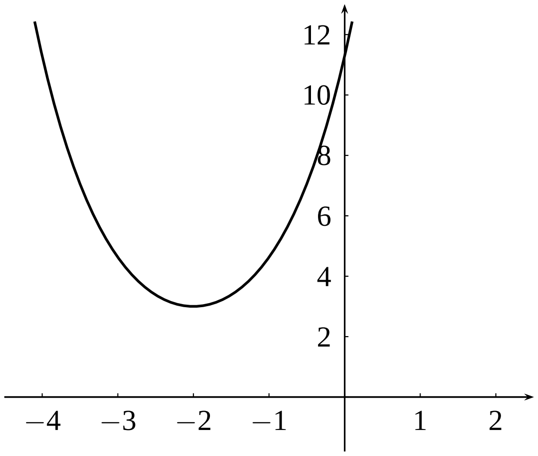
Solution: The domain of f is $\mathbb{R}$ and the range is $\mathbb{R}$. Put $y = 3 - \sinh x$. Then $3 - y = \sinh x$ and so $x = \sinh^{-1}(3-y) = f^{-1}(y)$. Interchanging x and y gives

$$f^{-1}(x) = \sinh^{-1}(3-x).$$

The domain and range of f^{-1} are both equal to $\mathbb{R}$.

4. Find the natural domain and the corresponding range of f, where $f(x) = 3\cosh(x+2)$. Sketch the graph of $f(x)$. Is this function injective ? If necessary, restrict the domain to obtain injectivity. Find an algebraic expression for the inverse function and sketch it.

Solution: The natural domain of f is $\mathbb{R}$ and, since $\cosh x \geq 1$ for all x, the range of f is $[3,\infty)$.

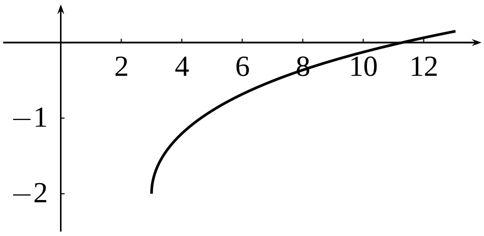


This function is not injective as it fails the horizontal line test (see graph above). If we restrict the domain to $[-2,\infty)$ we do have injectivity, and the range is still $[3,\infty)$.

Set $y = 3\cosh(x+2)$, then $y/3 = \cosh(x+2)$, and so $x+2 = \cosh^{-1}(y/3)$ and hence $x = \cosh^{-1}(y/3) - 2$. Switching x and y gives

$$f^{-1}(x) = \cosh^{-1}(x/3) - 2.$$

The domain of f^{-1} is $[3,\infty)$ and the range is $[-2,\infty)$.



5. Decide whether the following limits exist, and find them when they do exist. (You may find it helpful to factorise the expressions in the numerators and denominators.)

(a) $\lim\limits_{x\to 2} \dfrac{x^2-6x+8}{x^2-5x+6}$

Solution: The numerator and denominator are both zero when $x=2$, so are divisible by $x-2$. In fact $x^2-6x+8=(x-2)(x-4)$ and $x^2-5x+6=(x-2)(x-3)$. Hence,

$$\frac{x^2-6x+8}{x^2-5x+6}=\frac{x-4}{x-3}, \qquad x\neq 2,3,$$

and so

$$\lim_{x\to 2}\frac{x^2-6x+8}{x^2-5x+6}=\lim_{x\to 2}\frac{x-4}{x-3}=\frac{2-4}{2-3}=2.$$

(b) $\lim\limits_{x\to 1} \dfrac{x^3-3x+2}{x^2-2x+1}$

Solution: The numerator and denominator are each divisible by $x-1$, so

$$\frac{x^3-3x+2}{x^2-2x+1}=\frac{x^2+x-2}{x-1}, \qquad x\neq 1.$$

Since x^2+x-2 is also divisible by $x-1$, we find that

$$\frac{x^3-3x+2}{x^2-2x+1}=x+2, \qquad x\neq 1,$$

and so

$$\lim_{x\to 1}\frac{x^3-3x+2}{x^2-2x+1}=\lim_{x\to 1}(x+2)=3.$$

6. Find the following limits as $x\to\infty$. (Hint: divide top and bottom by the highest power of x in the denominator.)

(a) $\lim\limits_{x\to\infty} \dfrac{2x-3}{4x+5}$

Solution: Divide numerator and denominator by x:

$$\lim_{x\to\infty}\frac{2x-3}{4x+5}=\lim_{x\to\infty}\frac{2-(3/x)}{4+(5/x)}=\frac{\lim_{x\to\infty}\{2-(3/x)\}}{\lim_{x\to\infty}\{4+(5/x)\}}=\frac{2-0}{4+0}=\frac{2}{4}=\frac{1}{2}.$$

(b) $\lim\limits_{x\to\infty} \dfrac{5+x^2}{3+27x-x^3}$

Solution: Divide numerator and denominator by x^3 (highest power in denominator):

$$\lim_{x\to\infty}\frac{5+x^2}{3+27x-x^3}=\lim_{x\to\infty}\frac{(5/x^3)+(1/x)}{(3/x^3)+(27/x^2)-1}=\frac{0+0}{0+0-1}=0.$$

7. Use the squeeze law to find the following limits.

(a) $\lim\limits_{x\to 0} x^4 \cos\dfrac{1}{x}$

Solution: Since $-1 \leq \cos(1/x) \leq 1$, we have

$$-x^4 \leq x^4\cos(1/x) \leq x^4,$$

for all $x \neq 0$. Since $\lim\limits_{x\to 0} x^4 = 0$, the squeeze law implies that $\lim\limits_{x\to 0} x^4\cos(1/x)$ exists and is equal to zero.

(b) $\lim\limits_{x\to 0}\left(\sin x \sin\dfrac{1}{x}\right)$

Solution: Observe that if $x \neq 0$ then $\big|\sin x \sin(1/x)\big| = |\sin x| \cdot |\sin(1/x)| \leq |\sin x|$. Therefore,

$$-|\sin x| \leq \sin x \sin(1/x) \leq |\sin x|,$$

for all $x \neq 0$. (Note that we need the absolute value signs here because $\sin x$ is sometimes positive and sometimes negative.) Since $\lim\limits_{x\to 0} |\sin x| = 0$ the squeeze law implies that the limit $\lim\limits_{x\to 0}\{\sin x \sin(1/x)\}$ exists and equals zero.

8. Apply l'Hôpital's rule to evaluate the following limits.

(a) $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$.

Solution: Since $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to\infty} \sqrt{x} = \infty$ we can apply l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1}{x\left(\frac{1}{2}x^{-1/2}\right)} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

(b) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$.

Solution: Since $\lim\limits_{x\to 0} \tan x = 0$ and $\lim\limits_{x\to 0} x = 0$, we can apply l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = \sec^2 0 = 1.$$

Alternatively, since $\tan x = \sin x/\cos x$, we get

$$\lim_{x\to 0} \frac{\tan x}{x} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right) = 1\cdot 1 = 1.$$

(c) $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{2x}$;

Solution: The expression is of the form $\frac{0}{0}$ as $x \to 0$.

$$\lim_{x\to 0} \frac{e^{3x}-1}{2x} = \lim_{x\to 0} \frac{(d/dx)(e^{3x}-1)}{(d/dx)(2x)} = \lim_{x\to 0} \frac{3e^{3x}}{2} = \frac{3}{2}.$$

(d) $\lim\limits_{x\to 0} \dfrac{\cos 5x - 1}{x^2}$;

Solution: Note that the numerator and denominator are differentiable, and as $x \to 0$, $(\cos 5x - 1)/x^2$ has the indeterminate form $0/0$. The same is true after one application of l'Hôpital's rule. Hence l'Hôpital's rule applies twice and we get

$$\begin{aligned}\lim_{x\to 0} \frac{\cos 5x - 1}{x^2} &= \lim_{x\to 0} \frac{\frac{d}{dx}(\cos 5x - 1)}{\frac{d}{dx}(x^2)} = \lim_{x\to 0} \frac{-5\sin 5x}{2x} \\ &= \lim_{x\to 0} \frac{\frac{d}{dx}(-5\sin 5x)}{\frac{d}{dx}(2x)} = \lim_{x\to 0} \frac{-5^2\cos 5x}{2} = -\frac{25}{2}.\end{aligned}$$

(e) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$.

Solution: Note that the numerator and denominator are differentiable, and as $x \to 0$, $(1-\cos x)/x^2$ has the indeterminate form $0/0$. Hence l'Hôpital's rule applies twice and we get

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

(f) $\lim_{x\to 0} \dfrac{\tan 3x}{\tan 4x}$;

Solution: The expression is of the form $\frac{0}{0}$ as $x \to 0$.

$$\lim_{x\to 0} \frac{\tan(3x)}{\tan(4x)} = \lim_{x\to 0} \frac{3\sec^2(3x)}{4\sec^2(4x)} = \frac{3}{4}, \text{ since } \sec(0) = 1.$$

(g) $\lim_{x\to\infty} \dfrac{(\ln x)^3}{x^2}$;

Solution: This is an example of the ∞/∞ form of l'Hôpital's rule, and we have to apply this rule several times in succession.

$$\lim_{x\to\infty} \frac{(\ln x)^3}{x^2} = \lim_{x\to\infty} \frac{3(\ln x)^2}{x\cdot 2x} = \lim_{x\to\infty} \frac{6\ln x}{x\cdot 4x} = \lim_{x\to\infty} \frac{6}{x\cdot 8x} = 0.$$

(h) $\lim_{x\to 0} \dfrac{\sin 2x}{\sinh 3x}$;

Solution: The expression is of the form $\frac{0}{0}$ as $x \to 0$.

$$\lim_{x\to 0} \frac{\sin 2x}{\sinh 3x} = \lim_{x\to 0} \frac{2\cos 2x}{3\cosh 3x} = \frac{2}{3}.$$

(i) $\lim_{x\to 0} \dfrac{1-e^{-2x}}{\sin x}$;

Solution: The expression is of the form $\frac{0}{0}$ as $x \to 0$.

$$\lim_{x\to 0} \frac{1-e^{-2x}}{\sin x} = \lim_{x\to 0} \frac{2e^{-2x}}{\cos x} = 2.$$

(j) $\lim_{x\to 0} \dfrac{2x-\sin^{-1} x}{2x}$.

Solution: The expression is of the form $\frac{0}{0}$ as $x \to 0$.

$$\lim_{x\to 0} \frac{2x-\sin^{-1} x}{2x} = \lim_{x\to 0} \frac{2-(1-x^2)^{-1/2}}{2} = \frac{1}{2}.$$

(k) $\lim_{x\to -\infty} e^x \ln(x^2)$;

Solution: First we rewrite the function as a fraction so that we can apply l'Hôpital's rule. $e^x \ln(x^2) = \frac{\ln(x^2)}{e^{-x}}$. Since $\lim_{x\to -\infty} \ln(x^2) = \infty$ and $\lim_{x\to -\infty} e^{-x} = \infty$, l'Hôpital's rule may be used:

$$\lim_{x\to -\infty} \frac{\ln(x^2)}{e^{-x}} = \lim_{x\to -\infty} \frac{2/x}{-e^{-x}} = \lim_{x\to -\infty} -\frac{2}{xe^{-x}} = 0.$$

(l) $\lim_{x\to 0} \dfrac{\sqrt{4-x}-2}{x}$.

(The expression is of the form $\frac{0}{0}$ as $x \to 0$, so l'Hôpital's rule may be tried. Another method of solution is to rationalise the numerator.)

Solution: Applying l'Hôpital's rule gives $\lim_{x\to 0}\dfrac{\sqrt{4-x}-2}{x} = \lim_{x\to 0}\dfrac{-1}{2\sqrt{4-x}} = -\dfrac{1}{4}$.

Alternatively,

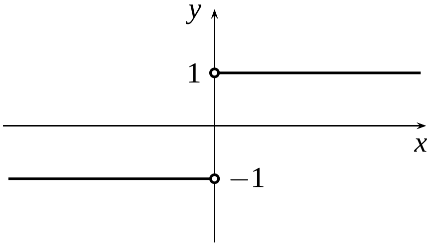
$$\lim_{x\to 0}\frac{\sqrt{4-x}-2}{x} = \lim_{x\to 0}\frac{(\sqrt{4-x}-2)(\sqrt{4-x}+2)}{x(\sqrt{4-x}+2)} = \lim_{x\to 0}\frac{-x}{x(\sqrt{4-x}+2)} = -\frac{1}{4}.$$

Questions for further practice

9. Write down the natural domain and sketch the graph of the functions given by the following rules. Then determine whether each function is injective:

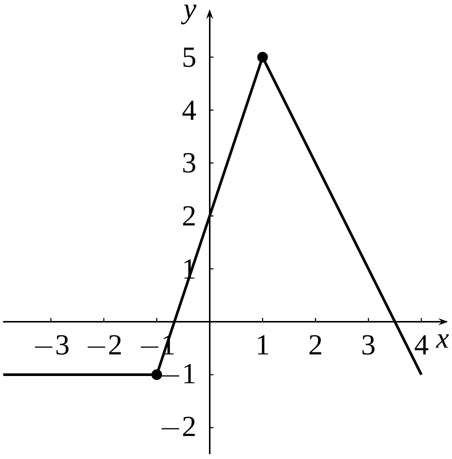
(a) $h(x) = \dfrac{x}{|x|}$

Solution: The domain is $\mathbb{R}\setminus\{0\}$ and the function fails (spectacularly!) to satisfy the horizontal line test, so is not injective.



(b) $k(x) = \begin{cases} -1 & \text{if } x \le -1, \\ 3x+2 & \text{if } |x| < 1, \\ 7-2x & \text{if } x \ge 1. \end{cases}$

Solution: The domain is $\mathbb{R}$ and the function is not injective.



10. Determine the natural domain and corresponding range for the two functions below. In each case find a formula for the inverse function $f^{-1}(x)$, after modifying the domain of f if necessary.

(a) $f(x) = (2+5x)^4$

Solution: The domain of f is $\mathbb{R} = (-\infty,\infty)$ and its range is $[0,\infty)$. However, f is not injective on the domain $\mathbb{R}$ since, for example, $f(0) = f(-\frac{4}{5}) = 16$. The graph of f is symmetrical about the vertical line $x = -\frac{2}{5}$ and is increasing for $x > -2/5$. Therefore if we restrict the domain to $[-2/5,\infty)$ the range will remain as $[0,\infty)$ but now we have an injective function. Put $y = (2+5x)^4$. Then $y^{1/4} = 2+5x$, so $x = (y^{1/4}-2)/5 = f^{-1}(y)$. Note that as $x \ge -2/5$, $y^{1/4}$ must be the *positive* fourth root of y. The inverse f^{-1} is given by

$$f^{-1}(x) = \frac{x^{1/4}-2}{5},$$

and has domain $[0,\infty)$ and range $[-2/5,\infty)$.

(b) $f(x) = \sqrt{e^x+1}$

Solution: The domain of f is $\mathbb{R} = (-\infty,\infty)$ and its range is $(1,\infty)$. The function is injective since it is an increasing function on its domain. (To see this, calculate the derivative $f'(x) = \frac{e^x}{2\sqrt{e^x+1}}$ and observe that this is always positive.)

Put $y = \sqrt{e^x+1}$. Then $y^2 = e^x+1$, so $e^x = y^2-1$ and $x = \ln(y^2-1) = f^{-1}(y)$. Switching x and y, we deduce that

$$f^{-1}(x) = \ln(x^2-1).$$

The domain of f^{-1} equals the range of $f(x)$, which is $(1,\infty)$. The range of $f^{-1}(x)$ equals the domain of $f(x)$, which is $\mathbb{R}$. (Note that the natural domain of $\ln(x^2-1)$ is in fact $\{x \in \mathbb{R} \mid |x| > 1\} = (-\infty,-1) \cup (1,\infty)$.)

11. In the following examples, assume that $\lim\limits_{x\to 1} f(x) = 3$, $\lim\limits_{x\to 1} g(x) = 4$ and $\lim\limits_{x\to 1} h(x) = 0$. Use the limit laws to decide whether these limits exist, and to find them when they do exist:

(a) $\lim\limits_{x\to 1} \dfrac{f(x)}{g(x)}$

Solution: The limit exists and is $\dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)} = \dfrac{3}{4}$.

(b) $\lim\limits_{x\to 1} \dfrac{f(x)h(x)}{g(x)+h(x)}$

Solution: Notice that

$$\lim_{x\to 1}\{f(x)h(x)\} = \{\lim_{x\to 1} f(x)\}\{\lim_{x\to 1} h(x)\} = 4\cdot 0 = 0,$$

$$\lim_{x\to 1}\{g(x)+h(x)\} = \lim_{x\to 1} g(x) + \lim_{x\to 1} h(x) = 4+0 = 4 \neq 0.$$

Therefore,

$$\lim_{x\to 1} \frac{f(x)h(x)}{g(x)+h(x)} = \frac{0}{4} = 0.$$

(c) $\lim\limits_{x\to 1}\{5g(x)+4h(x)\}$

Solution: The limit exists and is $5\lim_{x\to 1} g(x) + 4\lim_{x\to 1} h(x) = 5\cdot 4 + 4\cdot 0 = 20$.

(d) $\lim\limits_{x\to 1} \dfrac{g(x)}{h(x)}$

Solution: We cannot reach a definite conclusion from the information given in the question. For example, if we consider the functions $g(x) = 4$, a constant function, and $h(x) = x-1$, then $\lim\limits_{x\to 1} g(x) = 4$ and $\lim\limits_{x\to 1} h(x) = 0$ and the limit,

$$\lim_{x\to 1} \frac{g(x)}{h(x)} = \lim_{x\to 1} \frac{4}{x-1},$$

does not exist. This example has one-sided infinite limits, but not a two-sided limit. However, if we let $g(x) = 4$ and $h(x) = (x-1)^2$, then $\lim\limits_{x\to 1} g(x) = 4$ and $\lim\limits_{x\to 1} h(x) = 0$, but now we have that

$$\lim_{x\to 1} \frac{g(x)}{h(x)} = \lim_{x\to 1} \frac{4}{(x-1)^2} = +\infty.$$

12. Find the following limits at ±∞, if they exist:

(a) $\lim_{x\to\infty} \frac{8x^3+4x^2-5}{6x^3-7x+2}$

Solution: The limit exists and is given by

$$\lim_{x\to\infty} \frac{8x^3+4x^2-5}{6x^3-7x+2} = \lim_{x\to\infty} \frac{8+4/x-5/x^3}{6-7/x^2+2/x^3} = \frac{8+0-0}{6-0+0} = \frac{8}{6} = \frac{4}{3}.$$

(b) $\lim_{x\to-\infty} \frac{9x^2\sin x-3x+2}{6x^3-4x^2+5}$

Solution: The limit exists and is zero, because

$$\lim_{x\to-\infty} \frac{9x^2\sin x-3x+2}{6x^3-4x^2+5} = \lim_{x\to-\infty} \frac{(9/x)\sin x-3/x^2+2/x^3}{6-4/x+5/x^3} = \frac{0-0+0}{6-0+0} = 0.$$

(c) $\lim_{x\to-\infty} \frac{5x\sin x+7}{6x-11}$

Solution: This limit does not exist. To see this notice that the function can be rewritten in the form,

$$\frac{5x\sin x+7}{6x-11} = \frac{5\sin x+7/x}{6-11/x}.$$

When x is large, the right-hand side is approximately $(5/6)\sin x$. As this function takes every value in the interval $[-5/6, 5/6]$ as x runs through any interval of length 2π it follows that $(5/6)\sin x$ does not approach any *single* number as x goes to $-\infty$. So the required limit does not exist.

13. Investigate the following limits.

(a) $\lim_{x\to 0} \frac{1}{\sin x}$

Solution: The limit does not exist. As x approaches 0 along the positive half of the real line, $\sin x$ is small and positive, and so $1/\sin x$ is increasingly large and positive. But approaching 0 from the other side gives increasingly large and negative values for $1/\sin x$. Thus it has one-sided infinite limits, but no two-sided limit.

(b) $\lim_{x\to 0} \frac{1}{1-\cos x}$

Solution: Since $\cos x \leq 1$ for all x, the values of $1/(1-\cos x)$ are positive, where defined. Hence, the denominator tends to zero through positive values as $x \to 0$ from both sides. Hence, the limit is $+\infty$.

(c) $\lim_{x\to\infty} (3x-x^2)$

Solution: For x large, the quadratic term $-x^2$ grows faster than the linear term $3x$. Hence, the limit exists and equals $-\infty$. Another way to see this is to write $3x-x^2 = x(3-x)$. The first factor x tends to $+\infty$ while the second factor $3-x$ tends to $-\infty$. This is a case where the limit law for products can be applied to infinite limits, since the statement $(+\infty)(-\infty) = -\infty$ is correct and unambiguous. Hence the limit is $-\infty$.