THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 12

SS1001: Differential Calculus — Summer School, 2012

Web Page: `http://www.maths.usyd.edu.au/u/UG/SS/SS1001/`

Assumed knowledge

Basic algebra, differentiation, Taylor polynomials and complex numbers.

Objectives

By the end of Tutorial 12 you should be able to do the following:

12a understand the definition of a series and be able to tell when simple series, such as geometric series, converge and when they diverge;

12b know the Taylor series for functions such as e^x, $\cos x$ and $\sin x$, and be able to derive them;

12c be able to find the Taylor series for simple functions;

12d know that a Taylor series for a function $f(x)$ converges to $f(x)$ precisely when $\lim_{n\to\infty} R_n(x) = 0$;

12e be able to show when the Taylor series of simple functions converge;

12f be able to use the binomial series,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \ldots$$

and know that it converges for all $p \in \mathbb{R}$ when $x \in (-1,1)$;

12g know Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and how to derive it using Taylor series.

Preparatory questions

1. Which of the following series converge, and if so, to what?

(a) $0.999999... = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$

(b) $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots$

(c) $1 - 1 + 1 - 1 + 1 - \ldots$

(d) $1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \ldots.$

2. Find the Taylor series for $\cos(2x)$.

Questions to do in class

3. Show that if $x \in [-1, 1]$ then e^x can be approximated to within 0.025 by the polynomial

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

You may assume that $e < 3$.

4. (a) Find an expression for the n^{th} derivative of the function $\ln(1+x)$ about $x = 0$.

(b) Find the Taylor polynomial of order n for $\ln(1+x)$ about $x = 0$.

(c) Write down the formula for the Taylor series of $\ln(1+x)$ using sigma notation.

5. (a) Write down the Taylor polynomials about $x = 0$ of orders four and five for $\cos x$.

(b) Deduce that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + R_5(x)$, where $|R_5(x)| \leq \frac{x^6}{6!}$.

(c) Use part (**??**) to estimate $\cos 1$, and determine the accuracy of your estimate.

6. In this question we show how to obtain a Taylor series for $\sin^{-1}(x)$ without having to find a formula for the nth derivative of this function. We do this by using the binomial series, together with knowledge of the first derivative of $\sin^{-1}(x)$.

(a) Show that the binomial series expansion of $\frac{1}{\sqrt{1-x}}$ about the point 0 is

$$1 + \frac{1}{2}x + \frac{1 \cdot 3}{2^2 \cdot 2!}x^2 + \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}x^3 + \ldots = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}x^n.$$

(b) Hence find a series for $\frac{1}{\sqrt{1-x^2}}$.

(c) Find the first derivative of $y = \sin^{-1}(x)$.

(d) Find a series for $\sin^{-1}(x)$ by term-by-term integration.

Questions for further practice

7. (a) Find the n^{th} derivative of the function xe^{-x} by calculating the first few derivatives and establishing a pattern.

(b) Using part (a), compute $f^{(n)}(0)$, for all $n \geq 0$.

(c) Hence compute the Taylor series about $x = 0$ for xe^{-x}.

8. (a) Write down the Taylor series (about $x = 0$) for the function $\cos x$.

(b) Write down a series for $\cos\sqrt{x}$, $x \geq 0$.

(c) Show that $\cos(\sqrt{x}) = 1 - \frac{x}{2!} + \frac{x^2}{4!} + R_5(\sqrt{x})$, where $|R_5(\sqrt{x})| \leq \frac{x^3}{6!}$, $x \geq 0$.

(d) Show that $\left| \int_0^1 R_5(\sqrt{x})\, dx \right| < 0.0004$.

(e) Estimate $\int_0^1 \cos(\sqrt{x})\, dx$, correct to three decimal places.

9. (a) Write down the Taylor series for $\sinh x$ about $x = 0$.

(b) Find the remainder term $R_{2n+2}(x)$ for $\sinh x$.

(c) Given that $x^n/n! \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$, show that $\lim_{n\to\infty} R_{2n+2}(x) = 0$ for all $x \in \mathbb{R}$.

Short answers to selected exercises

Full solutions can be downloaded at the end of the week.

1. (a) The repeating decimal 0.999999... is a geometric series with leading term $\frac{9}{10}$ and common ratio $r = \frac{1}{10}$. The geometric series formula gives the sum as $\frac{9}{10}\frac{1}{1-\frac{1}{10}} = 1$.

(b) Geometric series with $r = 1/3$. Sum: $1/(1-r) = 3/2$.

(c) The sequence of partial sums $\{1,0,1,0,1,0, \ldots\}$ does not tend to a limit, and so the given series diverges.

(d) Recall that the Taylor series for e^x is $\sum_{n=0}^{\infty} x^n/n!$, which converges to e^x for all $x \in \mathbb{R}$. Therefore, $\sum_{n=0}^{\infty} 1/n! = e^1 = e$.

2. The Taylor series for $\cos x$ about $x = 0$ is $\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$, valid for all $x \in \mathbb{R}$. Replacing x by $2x$ gives the Taylor series for $\cos(2x)$: $\cos 2x = \sum_{n=0}^{\infty}(-1)^n \{(2)^{2n}/(2n)!\}x^{2n}$.

3. $e^x = P_4(x)+R_4(x)$, where $P_4(x) = 1+x+x^2/2!+x^3/3!+x^4/4!$ and $|R_4(x)| = |e^c x^5/5!| < 3/5! = 0.025$.

4. (a) Let $f(x) = \ln(1+x)$. Then $f(0) = 0$ and $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$, $n \geq 1$, which implies $f^{(n)}(0) = (-1)^{n-1}(n-1)!$.

(b) $P_n(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots + (-1)^{n-1}\dfrac{x^n}{n}$.

(c) $\ln(1+x) = \displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$.

5. (a) $P_4(x) = P_5(x) = 1 - x^2/2! + x^4/4!$.

(b) $R_5(x) = -\dfrac{\cos(c)}{6!}x^6$ and $|\cos(c)| \leq 1$ imply $|R_5(x)| \leq \dfrac{x^6}{6!}$.

(c) $\cos(1) = P_5(1)+R_5(1)$, where $P_5(1) = 13/24 = 0.54167$ and $|R_5(1)| \leq 1/6! = 0.00139$; a calculator gives $\cos(1) = 0.54030$ and so $R_5(1) = -0.00137$.

6. (a) In the formula for the binomial series for $(1+x)^p$, put $p = -1/2$ and replace x by $-x$.

(b) Replace x by x^2 and get $\dfrac{1}{\sqrt{1-x^2}} = 1 + \displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}x^{2n}$, $x \in (-1,1)$.

(c) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

(d) $\sin^{-1} x = \int (1-x^2)^{-1/2}\,dx$. Integrate the series in part (b) term by term, then evaluate the arbitrary constant by letting $x = 0$.

7. (a) Let $f(x) = xe^{-x}$. Then $f^{(n)}(x) = (-1)^n(x-n)e^{-x}$.

(b) $f^{(n)}(0) = (-1)^{n+1}n$.

(c) $xe^{-x} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{(n-1)!}$, same as $x \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$.

8. (a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. (b) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$. (e) 0.764.

9. (a) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

(b) $R_{2n+2}(x) = \frac{\cosh(c)}{(2n+3)!} x^{2n+3}, \; 0 < |c| < |x|$.

(c) $R_{2n+2}(x) \to 0$ as $n \to \infty$ because $1 < \cosh(c) < \cosh(x), \; x \neq 0$, and $|x|^{2n+3}/(2n+3)! \to 0$ as $n \to \infty$.