THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Tutorial 9

SS1001: Differential Calculus — Summer School, 2012

Web Page: `http://www.maths.usyd.edu.au/u/UG/SS/SS1001/`

Assumed knowledge

Partial derivatives; dot products of vectors; the chain rule.

Objectives

By the end of Tutorial 9 you should be able to:

9a know the definition of the gradient vector ∇f (or grad f) for general functions f of two variables, namely, $\nabla f(x,y) = f_x\,\mathbf{i} + f_y\,\mathbf{j}$, and calculate ∇f at both general and specific points;

9b interpret the directional derivative $D_{\mathbf{u}} f$ of a function $f(x,y)$ of two variables as the rate of change of f in the direction of the vector $\mathbf{u} = m\,\mathbf{i} + n\,\mathbf{j}$;

9c calculate the directional derivative of specific functions of two variables at a point (a,b) in specific and general directions;

9d interpret the directional derivative $D_{\mathbf{u}} f$ in the direction of a vector $\mathbf{u}$ as the dot product $\nabla f \cdot \hat{\mathbf{u}}$, where $\hat{\mathbf{u}} = \mathbf{u}/|\mathbf{u}|$ (the unit vector in the direction of $\mathbf{u}$);

9e recall that the maximum value of the directional derivative $D_{\mathbf{u}} f$ is attained when $\mathbf{u}$ is in the direction of the vector ∇f and that the value of $D_{\mathbf{u}} f$ is then $|\nabla f|$;

9f know that for functions $f(x,y)$ of two variables, when (a,b) is a point on the level curve $f(x,y) = k$, the vectors $\pm\nabla f(a,b)$ are perpendicular to the level curve at (a,b);

9g know that for functions $f(x,y)$ of two variables, the vectors $\pm(f_y\,\mathbf{i} - f_x\,\mathbf{j})$, which are perpendicular to ∇f, are tangential to the level curves of f, and are therefore directions in which the surface $z = f(x,y)$ is horizontal.

Preparatory questions

1. Find the directional derivative $D_{\mathbf{u}} f(x,y)$ if $f(x,y) = 2x^3 - 3xy + 5y^2$ and

(a) $\mathbf{u} = 3\,\mathbf{i} + 4\,\mathbf{j}$

(b) $\mathbf{u}$ is the unit vector $(\mathbf{i} + \sqrt{3}\,\mathbf{j})/2$.

Find $D_{\mathbf{u}} f(1,1)$ in each case.

2. Let $f(x,y) = 2x^3 + e^x \sin y$. Find $\nabla f(x,y)$ and calculate the value of $\nabla f(2,\pi)$.

Questions to do in class

3. Suppose that $\nabla f(x,y) = (2xy^2 - y)\,\mathbf{i} + (2x^2y - x + y^2)\,\mathbf{j}$. Find:

(a) the direction of steepest increase for $f(x,y)$ when $(x,y) = (1,1)$;

(b) the magnitude of the steepest increase when $(x,y) = (1,1)$;

(c) the direction of steepest decrease for $f(x,y)$ when $(x,y) = (1,1)$;

(d) the directions in which the surface $z = f(x,y)$ is horizontal at $(x,y) = (1,1)$.

4. If $f(x,y) = x - y^2$, find $\nabla f(3,-1)$ and use it to find the equation of the tangent line to the level curve $f(x,y) = 2$ at the point $(3,-1)$. Sketch the level curve, the tangent line and the gradient vector.

5. Suppose that you are climbing a 1000 metre hill whose shape is given by the equation,

$$z = 1000 - 0.01x^2 - 0.02y^2,$$

and you are standing at the point with coordinates $(60, 100, 764)$. Suppose also that the positive x-axis points East and the positive y-axis points North.

(a) In which direction should you face initially in order to reach the top of the hill via the steepest route?

(b) If you climb in that direction, at what angle above the horizontal will you be climbing initially?

(c) In which direction should you travel initially if you want to maintain your height above sea level at 764 metres?

6. (a) Find the directional derivative of $f(x,y) = \sqrt{5x-4y}$ at the point $(4,1)$ in the direction given by $\sqrt{3}\,\mathbf{i} - \mathbf{j}$.

(b) Find the direction of steepest slope at the point $(4,1)$, and the slope in that direction.

Questions for further practice

7. Let $f(x,y) = x^3 - \sqrt{y^2+8x}$.

(a) Find the directional derivative of f at the point $(2,3)$ in the direction which makes an angle of $-\pi/4$ with the positive direction of the x-axis.

(b) Find the value and direction of the maximum directional derivative of f at the point $(2,3)$.

8. Find the directions in which the directional derivative of $f(x,y) = x^2 + \sin xy$ at $(1,0)$ has value 1.

9. Suppose that for a function $z = f(x,y)$ and a point (a,b), we have $f_x(a,b) = 1$ and $f_y(a,b) = -2$.

(a) In which direction(s) $\mathbf{u}$ is $D_{\mathbf{u}} f(a,b) = 0$?

(b) In which direction $\mathbf{u}$ is $D_{\mathbf{u}} f(a,b)$ largest?

10. Calculate dy/dx when y is defined implicitly as a function of x by $\sin(xy) + y^2e^{-x} = x+4$. What is the slope of the tangent to the curve at the point $(0,2)$?

Short answers to selected exercises

Full solutions can be downloaded at the end of the week.

1. $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} = (6x^2-3y)\,\mathbf{i} + (10y-3x)\,\mathbf{j}$. At $(1,1)$, $\nabla f = 3\,\mathbf{i} + 7\,\mathbf{j}$.

(a) $\hat{\mathbf{u}} = (3\,\mathbf{i} + 4\,\mathbf{j})/5$: $D_{\mathbf{u}} f = \nabla f \cdot \hat{\mathbf{u}} = (18x^2 - 12x + 31y)/5$. At $(1,1)$, $D_{\mathbf{u}} f = 37/5$.

(b) $\hat{\mathbf{u}} = (\mathbf{i} + \sqrt{3}\,\mathbf{j})/2$: $D_{\mathbf{u}} f = \{6x^2 - 3\sqrt{3}x + (10\sqrt{3}-3)y\}/2$. At $(1,1)$, $D_{\mathbf{u}} f = (3+7\sqrt{3})/2$.

2. $\nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} = (6x^2 + e^x \sin y)\,\mathbf{i} + e^x \cos y\,\mathbf{j}$. At $(2,\pi)$, $\nabla f = 24\,\mathbf{i} - e^2\,\mathbf{j}$.

3. (a) $\nabla f(1,1) = \mathbf{i} + 2\,\mathbf{j}$. (b) $|\nabla f(1,1)| = \sqrt{5}$. (c) $-(\mathbf{i} + 2\,\mathbf{j})$. (d) $\pm(2\,\mathbf{i} - \mathbf{j})$.

4. $\nabla f(3,-1) = \mathbf{i} + 2\,\mathbf{j}$. Equation of tangent line: $x + 2y = 1$.

5. (a) Direction (facing horizontally) of steepest ascent is $\nabla z(60,100) = -(1.2\,\mathbf{i} + 4\,\mathbf{j})$, 73.3 degrees south of west.

(b) Slope $|\nabla z| = 4.176$, angle 76.5° above the horizontal (very steep).

(c) Direction $\pm(10\,\mathbf{i} - 3\,\mathbf{j})$, 16.7° south of east or north of west.

6. (a) $D_{\mathbf{u}} f(4,1) = (5\sqrt{3}+4)/16.$ (b) $8\nabla f = 5\,\mathbf{i} - 4\,\mathbf{j}$, $|\nabla f| = \sqrt{41}/8.$

7. (a) $D_{\mathbf{u}} f(2,3) = 59/(5\sqrt{2}).$ (b) $|\nabla f| = \sqrt{3145}/5$, $56\,\mathbf{i} - 3\,\mathbf{j}$.

8. Two directions: $\mathbf{u} = \mathbf{j}$ and $\mathbf{u} = (4\,\mathbf{i} - 3\,\mathbf{j})/5$ or $4\,\mathbf{i} - 3\,\mathbf{j}$.

9. (a) $\pm(2\,\mathbf{i} + \mathbf{j})$. (b) $\mathbf{i} - 2\,\mathbf{j}$.

10. $\dfrac{dy}{dx} = -\dfrac{y\cos(xy) - y^2 e^{-x} - 1}{x\cos(xy) + 2ye^{-x}}$; at $(0,2)$, $\dfrac{dy}{dx} = \dfrac{3}{4}$.