CHAPTER 1

Numbers and sets

Ask any person in the street what mathematics is about and their answer will almost certainly be “numbers”. In fact mathematics also includes the study of logic and structure and geometry, but ideas about numbers, real numbers in particular, are fundamental to calculus and many other branches of mathematics.

In this chapter we review and extend previous work on numbers and sets, particularly the set of real numbers, before introducing the set of complex numbers.

1.1 Different types of numbers

Our understanding of numbers, what they are and how they work, develops from simple counting through fractions and negative numbers to an appreciation of irrational numbers and real numbers. Mathematically, different types of numbers belong to different sets:

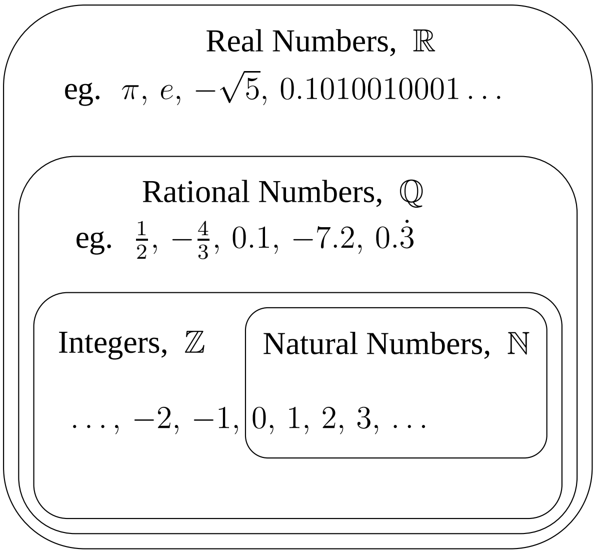
The set of natural numbers $\{0, 1, 2, 3, 4, \ldots\}$, which we denote by the symbol $\mathbb{N}$. It is *closed* under the operations of addition and multiplication. That is, adding two positive integers gives another positive integer, as does multiplying them together.

The set of integers $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$, denoted by $\mathbb{Z}$ is the set of whole numbers, including both positive whole numbers, negative whole numbers and zero. The set of integers is closed under the operations of addition, subtraction and multiplication.

The set of rational numbers, denoted by $\mathbb{Q}$, is the set of all numbers of the form n/m where n and m are integers and $m \neq 0$. Some examples are $\frac{1}{2}, -\frac{1}{4}, \frac{4}{3}$. Rational numbers include decimals which either terminate or repeat. Note that the integers are a subset of the rational numbers, since they are of the form n/m where $m = 1$. The set of rational numbers is closed under the operations of addition, subtraction, multiplication and division, provided that division by zero is excluded.

The set of real numbers, denoted by $\mathbb{R}$. This includes all rational numbers and all irrational numbers. Irrational numbers cannot be expressed as n/m, where m and n are integers, although some may be interpreted geometrically. For example, $\sqrt{2}$ is the length of a diagonal of a unit square. The irrational number π is the ratio of the length of the circumference of a circle to the circle’s diameter.

The set of real numbers, $\mathbb{R}$, contains all the other number sets. In fact we can summarise the numbers sets diagrammatically



1.2 Sets and notation

Set notation is a convenient and precise way to write about collections of objects. This type of notation is widely used in mathematics, both in this course and in other units of study.

Definition
A set is a collection of objects which are called members or elements.

A set can be denoted by its name, for example the set of real numbers $\mathbb{R}$, or written as a list, for example $\{a, b, c, d\ \}$. When sets are written as a list of elements, each element is separated from the others in the list by a comma and the whole list is surrounded by braces. Three dots may be used to mean "and so on" if the list of elements is large; for example $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Some useful set notation

Correct mathematical notation conveys information accurately and concisely. Here is some notation that is used with sets.

The symbol $\in$ means "is an element of". For example $-3 \in \mathbb{Z}$ should be read as "-3 is an element of the set of integers"; $y \in B$, reads "y is an element of the set B" or "y is a member of the set B".

The symbol $\subseteq$ should be read as "is a subset of" For example $\mathbb{N} \subseteq \mathbb{Z}$, is read as "the set of natural numbers is a subset of the set of integers" or "the set of natural numbers is contained in the set of integers". Sometimes you may see the symbol $\subset$ which means that the smaller set is strictly a subset of the larger; the two cannot be equal. For example, it is most precise to write $\mathbb{N} \subset \mathbb{Z}$ as the two sets can never be the same. The reversed symbol $\supseteq$ means "contains" For example $\mathbb{R} \supseteq \mathbb{Q}$, reads "the set of real numbers contains the set of rational numbers". There is also a symbol $\supset$ which means "contains, but is not equal to". If $A \subseteq B$ then $B \supseteq A$.

A forward slash through any of these symbols above means "not". For example, $-1 \notin \mathbb{N}$ says in words "-1 is not an element of the set of natural numbers". Another example, $\mathbb{R} \not\subseteq \mathbb{Z}$, is read as "the set of real numbers is not a subset of the set of integers."

There are other symbols which describe sets formed from other sets.

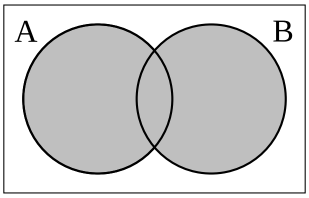
The expression $A \cup B$ denotes the union of set A with set B. The union of two sets is the set of elements which occur in either one or both of the sets. If an element occurs in both sets, it is only listed once in the union. For example $\{1, 2, 3, 4, \} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$.

The intersection of sets A and B is written $A \cap B$. The intersection of two sets is the set of elements which occur in *both* of the sets. For example $\{1, 2, 3, 4, \} \cap \{3, 4, 5, 6\} = \{3, 4\}$.

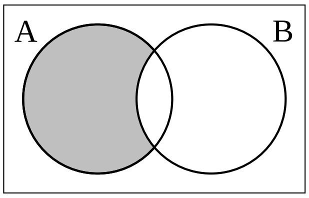
The symbol $\backslash$, which is read "minus", is used to indicate the set of elements which are in one set but not in another. That is, $A\backslash B$ is the set of all elements which are in A but not in B. So for example, $\{1, 2, 3, 4, \}\backslash\{3, 4, 5, 6\} = \{1, 2\}$.

A set can be represented in a simple, graphical way by a Venn diagram. Each set is drawn as a circle, a square or some other closed shape. Shapes representing sets may overlap one another if sets have elements in common. Sometimes, the elements of the sets are written on the Venn diagram but often they are not. Different parts of a Venn diagram can be shaded to illustrate different parts of the set.

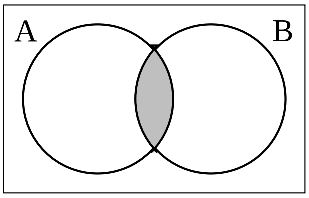
Venn diagrams are a useful way to represent relations between sets. Note that $A\backslash B$ is not the same as $B\backslash A$.



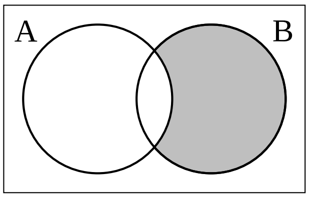
$A \cup B$



$A\backslash B$



$A \cap B$



$B\backslash A$

If we want to specify a set whose elements fulfil a certain condition then we do this in the following way:

Examples 1.2a

i) $$A = \{x \in \mathbb{Q} \,|\, x > 0\}$$

In words, this says "A is the set of all rational numbers x such that x is positive". Alternatively " A is the set of all positive rational numbers". The vertical slash should be read as "such that".

ii) Let W be the set of words in English. Then

$$B = \{x \text{ in } W \,|\, x \text{ begins with the letter "P"}\}$$

reads "B is the set of all elements x of the set of English words such that x begins with P" or "B is the set of all English words that begin with P."

iii) $$C = \{x \in \mathbb{Z} \,|\, \tfrac{x}{2} \in \mathbb{Z}\}$$

In words this says "C is the set of all integers x such that $x/2$ is an integer" or "C is the set of all even integers".

iv) $$D = \{x \in \mathbb{R} \,|\, -1 < x \leq 1\}$$

This reads "D is the set of all real numbers which are greater than -1 and less or equal to 1" ♢

It is worth taking the time to convert these sorts of expressions into words when you come across them. Mathematical notation is both precise and terse. To use information presented in this way well, you need to understand exactly what the notation means.

1.3 Intervals and the real number line

Interval notation

Sets of real numbers which lie between two end points, such as the last example, can be represented using interval notation. For example

$$B = \{x \in \mathbb{R} \,|\, -1 < x \leq 1\} = (-1, 1]$$

A *curved bracket* is used to show that an endpoint (such as -1 in the example) is not included in the set and a *square bracket* is used when the endpoint is part of the set.

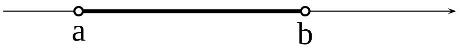
The real number line

Every real number can be located on the real number line. For example:

It is straightforward to sketch sets that are written using interval notation on the real number line.

Note that an open dot is used if the end point of the interval is not included in the set. If the endpoint is part of the set, then a closed dot is used.

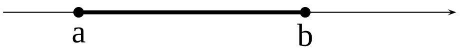
An interval where neither endpoint is part of the set is called an open interval.



$$(a,\, b) = \{x \in \mathbb{R}\,|a < x < b\}$$

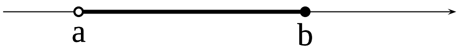
The interval $(a, b) = \{x \in \mathbb{R}\,|a < x < b\}$ and the point (a, b) in the Cartesian plane are written in exactly the same way. They are not, however, the same thing. It is usually clear from the context whether (a, b) represents a point or an interval.

If both endpoints are part of the interval it is called a closed interval.



$$[a,\, b] = \{x \in \mathbb{R}\,|a \le x \le b\}$$

It is also possible that one endpoint will be in the set and the other will not be. For example,



$$(a,\, b] = \{x \in \mathbb{R}\,|a < x \le b\}$$

$$[a,\, b) = \{x \in \mathbb{R}\,|a \le x < b\}$$

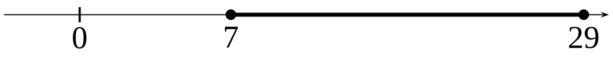
There is special notation for sets of the number line that extend infinitely in one direction or the other.

$$(a, \infty) = \{x \in \mathbb{R}\,|x > a\}; \quad (-\infty, a) = \{x \in \mathbb{R}\,|x < a\}$$
$$[a, \infty) = \{x \in \mathbb{R}\,|x \ge a\}; \quad (-\infty, a] = \{x \in \mathbb{R}\,|x \le a\}$$

Note that ∞ is *not* a number, rather, it represents infinity. Both ∞ and $-\infty$ always take a round bracket. Here are some other examples:

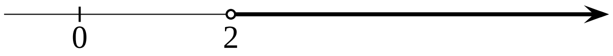
Examples 1.3a

i) $A = [7, 29] = \{x \in \mathbb{R}\,|7 \le x \le 29\}$



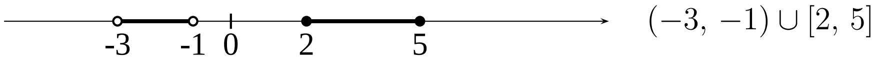
$[7,\, 29]$

ii) $S = (2, \infty) = \{x \in \mathbb{R}\,|x > 2\}$

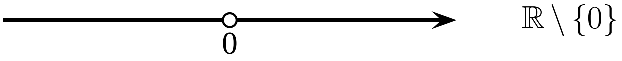


$(2,\, \infty)$

iii) $V = (-3, -1) \cup [2, 5] = \{x \in \mathbb{R} \,|\, -3 < x < -1 \text{ or } 2 \le x \le 5\}$



iv) $T = (-\infty, 0) \cup (0, \infty) = \{x \in \mathbb{R} \,|\, x \ne 0\} = \mathbb{R} \setminus \{0\}$. As you can see there may be a number of ways of writing down a set.



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The **modulus** or **absolute value** $|x|$ of a real number x gives the distance on the real number line from x to zero. The modulus of x is defined in this way:

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0. \end{cases}$$

For example $|5| = 5$ and $|-10| = 10$.

The distance between two numbers on the number line can also be expressed using modulus. The distance between x and y is given by $|x - y| = |y - x|$. For example, the distance between 3 and -4 is $|3 - (-4)| = |3 + 4| = 7$ which is what we intuitively expect to be the distance from -4 to 3. Alternatively we could have written $|-4 - 3| = |-7| = 7$.

1.4 Imaginary numbers

Suppose that you are asked to solve the equation

$$x^2 + 1 = 0.$$

Your first response might be to say that there will be two solutions as it is a quadratic equation. Very quickly you might write down the line;

$$x^2 = -1.$$

At that point you might conclude, correctly, that there are no real solutions to the equation. But what if we agree that there exists a number which is the square root of -1? Then we could write:

$$x = \pm\sqrt{-1}.$$

Such a number does indeed exist, although it is not a real number, and is known as an **imaginary number**. We denote it by i (although some branches of engineering use j instead). Since $i = \sqrt{-1}$ we can write

$$i^2 = (\sqrt{-1})^2 = -1 = (-i)^2.$$

and so the equation $x^2 + 1 = 0$ now has two imaginary solutions, namely i and $-i$.

What about the equations

$$x^2 + 4 = 0 \text{ and } y^2 + 7 = 0?$$

The first of these has imaginary solutions $x = \pm\sqrt{-4} = \pm 2\sqrt{-1} = \pm 2i$; the second has solutions $y = \pm\sqrt{7}i$.

Any non–zero real multiple of i is an imaginary number. The square of an imaginary number is a negative number. For example $3i$, $-20i$, $-i/5$, $0.125i$ and πi are all imaginary numbers, and $(3i)^2 = -9$, $(-20i)^2 = -400$, $(-i/5)^2 = -1/25$ and so on.

1.5 Complex numbers and solutions to quadratic equations

Now, suppose that you are given this equation to solve:

$$x^2 - 4x + 5 = 0.$$

Using the quadratic formula:

$$x = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm \sqrt{-1} = 2 \pm i.$$

These solutions are not purely imaginary, although they do involve an imaginary number. The solutions $2 + i$ and $2 - i$ are **complex numbers**. They have a real part and an imaginary part. For example, the real part of $2 + i$ is 2; we write $\mathrm{Re}(2 + i) = 2$. The imaginary part of a complex number is the coefficient of i, so the imaginary part of $2+i$ is 1 or $\mathrm{Im}(2+i) = 1$. You ✍ may like to show by substitution that $2 + i$ and $2 - i$ are indeed solutions of $x^2 - 4x + 5 = 0$.

Examples 1.5a

i) Consider the complex number $3 + 8i$. Re$(3 + 8i) = 3$ and Im$(3 + 8i) = 8$.

ii) If $z = 1/2 - 5i$ then Re$(z) = \frac{1}{2}$ and Im$(z) = -5$.

iii) For the purely imaginary number $-7i$, Re$(-7i) = 0$ and Im$(-7i) = -7$.

iv) For the real number 4 (which, of course is also a complex number) Re$(4) = 4$ and Im$(4) = 0$. ♢

If we allow complex numbers as solutions to quadratic equations then every quadratic equation will always have solutions, either both real or both complex.

We can see this in general if we look at the quadratic formula. The solution to the quadratic equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Whether $ax^2 + bx + c = 0$ has (purely) real or complex roots depends on the expression $b^2 - 4ac$ which is known as the discriminant.

$$x \begin{cases} \text{is real} & \text{if } b^2 - 4ac \geq 0 \\ \text{is complex} & \text{if } b^2 - 4ac < 0. \end{cases}$$

Example 1.5b The solutions to $x^2+6x+25=0$ must be complex since $b^2-4ac=-64<0$. Using the quadratic formula, the solutions are found to be $-3+4i$ and $-3-4i$. These solutions are complex conjugates of each other. This will be examined further in the next chapter. ◊

1.6 Arithmetic with complex numbers

Complex numbers can be added or multiplied together, subtracted one from the other or divided by one another.

Consider two complex numbers: $z=a+bi$ and $w=c+di$. Here the real part of z is a and the imaginary part of z is b. The real part of w is c and the imaginary part of w is d.

Addition

$$\begin{aligned} z+w &= (a+bi)+(c+di) \\ &= (a+c)+(b+d)i \end{aligned}$$

Rule: Add real parts to real parts and imaginary parts to imaginary parts.

Example 1.6a

$$\begin{aligned} (3-4i)+(1+2i) &= 3+1+(-4+2)i \\ &= 4-2i \end{aligned}$$

◊

Subtraction

$$\begin{aligned} z-w &= (a+bi)-(c+di) \\ &= (a-c)+(b-d)i \end{aligned}$$

Rule: Subtract real parts from real parts and imaginary parts from imaginary parts.

Example 1.6b

$$\begin{aligned} (3-4i)-(1+2i) &= 3-1+(-4-2)i \\ &= 2-6i \end{aligned}$$

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Multiplication

$$\begin{aligned} zw &= (a+bi)(c+di) \\ &= ac+adi+bci+(bd)i^2 \\ &= (ac-bd)+(ad+cb)i \end{aligned}$$

Rule: Expand the brackets in the normal way, remembering that i^2 can be simplified to -1 and collect terms into real and imaginary parts.

Example 1.6c

$$\begin{aligned}(3-4i)(1+2i) &= 3-4i+6i-8i^2\\ &= 3+2i+8\\ &= 11+2i\end{aligned}$$

$\Diamond$

To divide one complex number by another we need to know about the complex conjugate.

Definition The complex conjugate of the complex number $z = a + ib$ is $\overline{z} = a - ib$.

Notice that if $z = a + ib$ then $z\bar{z} = \bar{z}z = a^2 + b^2$. In particular, $z\bar{z}$ is always a non–negative real number and $z\bar{z} = 0$ if and only if $z = 0$. This observation is exactly what we need when dividing one complex number by another non–zero complex number.

Examples 1.6d

i) $\overline{3+5i} = 3-5i$

ii) $\overline{2-7i} = 2+7i$

iii) If z is a real number then $\overline{z} = z$. If z is a purely imaginary number then $\overline{z} = -z$. For example $\overline{3i} = -3i$. $\Diamond$

Division If $w \neq 0$ then to find $\frac{z}{w}$ we multiply both top and bottom by the complex conjugate of w.

$$\begin{aligned}\frac{z}{w} &= \frac{z}{w}\frac{\overline{w}}{\overline{w}}\\ &= \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)}\\ &= \frac{ac-adi+cbi-(bd)i^2}{c^2-cdi+cdi-d^2i^2}\\ &= \frac{(ac+bd)+(cb-ad)i}{c^2+d^2}\end{aligned}$$

This process is similar to rationalising the denominator of a quotient of surds. Multiplying by the complex conjugate of the divisor produces a real number in the denominator and allows the number to be written in the form $a + bi$.

Example 1.6e

$$\begin{aligned}\frac{5-10i}{1+2i} &= \frac{(5-10i)(1-2i)}{(1+2i)(1-2i)}\\ &= \frac{5-20i+20i^2}{1-2i+2i-4i^2}\\ &= \frac{-15-20i}{5}\\ &= -3-4i\end{aligned}$$

◊

Equality

Two complex numbers are equal to each other if and only if both their real and imaginary parts are equal. In other words, if $z = a + bi$ and $w = c + di$, then $z = w$ if and only if $a = c$ and $b = d$.

1.7 The set of complex numbers

We can think of a real number as a particular type of complex number, one with zero imaginary part. The complex numbers include real numbers and form a set which encompasses the set of real numbers and hence all of the other number sets.

Definition The set of complex numbers $\mathbb{C}$ is the set of all numbers of the form $a + ib$ where a and b are real numbers and $i^2 = -1$.

Alternatively, we could write:

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}, i^2 = -1\}.$$

We also have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}\,.$$

The complex numbers, like the set of real numbers, is closed under addition, subtraction, multiplication and division.

Complex numbers, however, lack an important property of the real numbers.

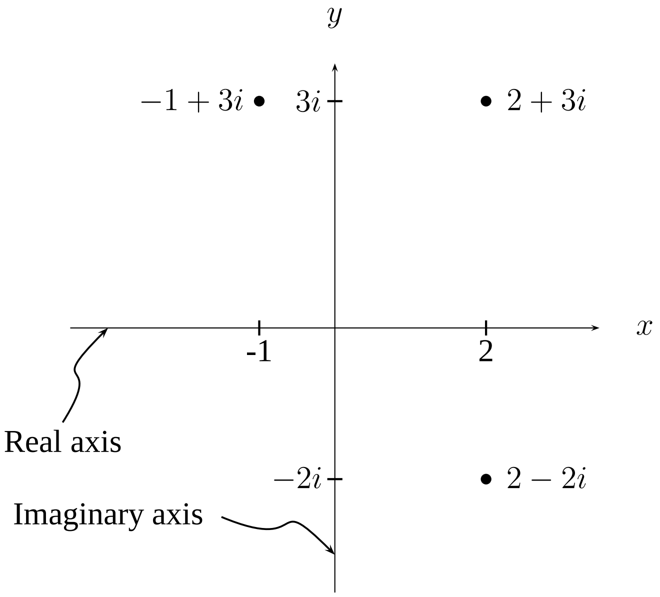
The set of real numbers is ordered; that is, if we have any two real numbers x and y we can say that either $x > y$ or $x < y$ or $x = y$. One of these alternatives will always be true. Because of this property we are able to represent real numbers on the real number line.

The set of complex numbers is not ordered. Consider two complex numbers: $2 - 3i$ and $-1+5i$. Clearly $2-3i \neq -1+5i$ as neither their real nor their imaginary parts are the same. But it makes no sense to write $2 - 3i > -1 + 5i$ or $2 - 3i < -1 + 5i$. It *does* make sense to write $\mathrm{Re}(2 - 3i) > \mathrm{Re}(-1 + 5i)$ and $\mathrm{Im}(2 - 3i) < \mathrm{Im}(-1 + 5i)$ but this is because the real part and the imaginary part of a complex numbers are both real numbers.

Because the set of complex numbers is not ordered, complex numbers cannot be represented as points on a line. Instead, complex numbers are represented as points on the complex plane.

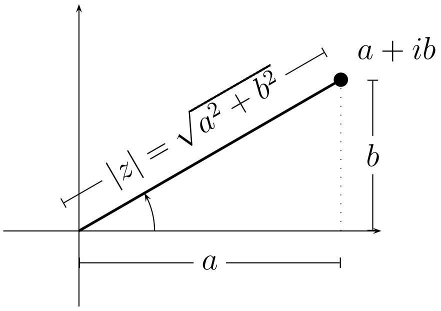
The Complex Plane

The complex plane or Argand plane allows complex numbers to be represented graphically. The horizontal axis in the complex plane is called the real axis. All real numbers lie on the horizontal axis in the complex plane; positive numbers to the right of the origin, negative numbers to its left. The vertical axis is known as the imaginary axis. All purely imaginary numbers lie on the vertical axis. Each point in the complex plane corresponds to a single complex number. This is a little different to the Cartesian plane used in coordinate geometry, where each point corresponds to an ordered pair of real numbers. For example:



The modulus of a complex number

For a real number x, the modulus of x, written as $|x|$, gives the distance on the real number line from x to the origin at zero. For a complex number $z = a + bi$, the modulus of z, written $|z|$, gives the distance in the complex plane from z to the origin.



If $z = a + bi$ then geometrically, by Pythagoras' Theorem,

(1.7a) $$|z| = \sqrt{a^2 + b^2}.$$

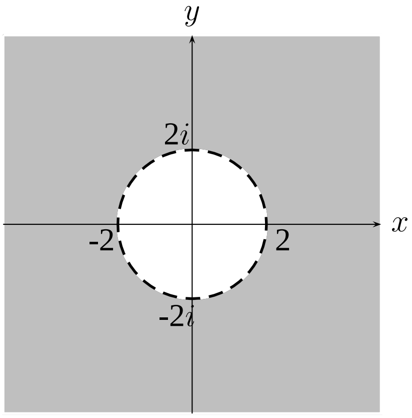
Alternatively we can use the complex conjugate of z to find $|z|$. Since $z\overline{z} = (a+bi)(a-bi) = a^2 + b^2$ we can write $|z| = \sqrt{z\overline{z}}$.

The modulus of a complex number is a real number and so it makes sense to write something like $|1 + i| < |2 - 3i|$; however two complex numbers with the same modulus need not be equal. For example $|4 - 3i| = |1 + 2\sqrt{6}i| = 5$. Note that the modulus is always a positive real number or zero.

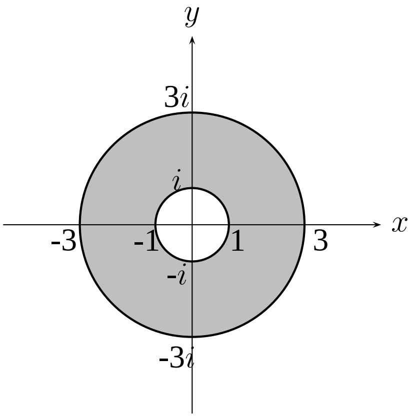
The modulus can be used to specify subsets of the complex numbers, which can be graphed in the complex plane.

Examples 1.7b

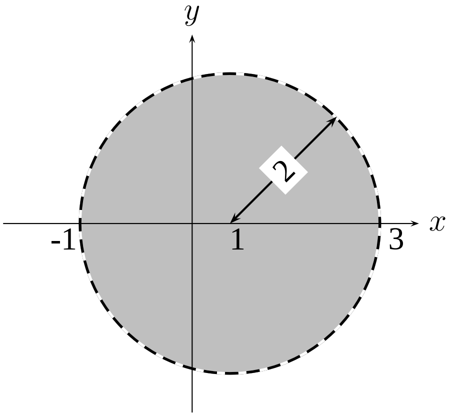
i) $\{z \in \mathbb{C} \,|\, |z| > 2\}$ is the set of complex numbers z such that z is more than 2 units distant from the origin.



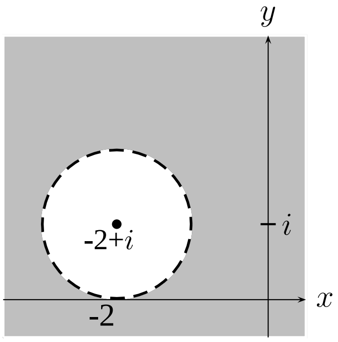
ii) $\{z \in \mathbb{C} \,|\, 1 \le |z| \le 3\}$ is the set of complex numbers which are between one and three units distant from the origin.



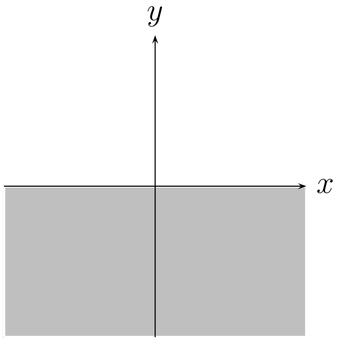
iii) $\{z \in \mathbb{C} \,|\, |z-1| < 2\}$. As with real numbers, $|z-1|$ is exactly the distance from z to 1. Hence, this is the set of all complex numbers whose distance from 1 is less than 2. Geometrically, these are all points in the complex plane that are inside the circle, centre $(1, 0)$, radius 2.



iv) $\{z \in \mathbb{C} \,|\, |z+2-i| > 1\}$ Here $|z+2-i| = |z-(-2+i)|$ is the distance from a complex number z to $-2+i$. So this set is the set of all complex numbers whose distance from $-2+i$ is greater than one unit. In other words, this is the set of points in the complex plane with are strictly outside the circle of radius 1 and centre $-2+i$.



v) Here is a different type of subset of the complex numbers: $\{z \in \mathbb{C} \,|\mathrm{Im} z \leq 0\}$ is the set of all complex number whose imaginary part is less or equal to zero.



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In the next chapter we will explore some uses of the complex plane representation of complex numbers and show how an understanding of the geometry of complex numbers is useful in performing certain types of calculations.

Exercises

In addition to doing the following exercises you should look at the on–line quiz

`www.maths.usyd.edu.au/u/UG/JM/MATH1001/Quizzes/quiz1.html`

which covers the material in this chapter. You can get to this page from the course homepage.

1.1 In each of the following perform the indicated operations and give the final answer in the form $x + iy$:

a) $(5 - 2i) + (2 + 3i)$
b) $(2 - i) - (6 - 3i)$
c) $(2 + 3i)(-2 - 3i)$
d) $-i(5 + i)$
e) $1/i$
f) $(a + ib)(a - ib)$
g) $6i/(6 - 5i)$
h) $(a + ib)/(a - ib)$
i) $1/(3 + 2i)$
j) $i^2, i^3, i^4, \ldots, i^{10}$
k) $(1 + i)/(1 - i)$
l) $[i/(1 - i)] + [(1 - i)/i]$
m) $(1/i) - 3i/(1 - i)$
n) $i^{123} - 4i^9 - 4i$

1.2 Simplify the following expressions:

a) $\operatorname{Im} \dfrac{1}{1 + i}$
b) $\operatorname{Re} \dfrac{(1 - i)^2}{1 + 2i}$
c) $|\cos\theta + i\sin\theta|$, where θ is any angle
d) $\left|\dfrac{1 + 3i}{3 + i}\right|$
e) $\left|\dfrac{(1 + i)^6}{i^3(1 + 4i)^2}\right|$.

1.3 If $z = x + iy$, express each of the following explicitly in terms of x and y:

a) $\mathrm{Re}\,(z/\overline{z})$
b) $|(z/\overline{z})|$
c) $\mathrm{Im}\,z^3$
d) $\mathrm{Re}\ z^4$
e) $|z^6|$
f) $|(z+1)/(z-1)|$
g) $\mathrm{Re}\,(1/z^2)$.

1.4 If $z = 5 + 12i$ and $w = 3 + 4i$, express $w + z$, $z - w$, zw and z/w in the form $a + ib$. Use these results to verify that

a) $|zw| = |z||w|$
b) $|z/w| = |z|/|w|$
c) $|z + w| \leq |z| + |w|$
d) $|z - w| \geq |z| - |w|$.

1.5 Show that for any complex number z, $|\overline{z}| = |z|$.

1.6 If $\overline{z} = z$, what can you say about z?

1.7 If $z = 3 - 2i$, plot z, $-z$, $\overline{z}$ and $-\overline{z}$ as points in the complex plane.

1.8 Give a geometric justification of the *triangle inequality*:

$$|z_1 + z_2| \leq |z_1| + |z_2| \,,$$

where z_1 and z_2 are any two complex numbers.

1.9 In each of the following cases, find the locus of points in the complex plane satisfying the given relation (describe the locus, sketch it, and give its cartesian equation):

a) $\mathrm{Im}\,z \geq 0$
b) $0 < \mathrm{Im}(z + 1) \leq 2\pi$
c) $-1 \leq \mathrm{Re}\ z < 1$
d) $\mathrm{Re}(iz) = 3$
e) $\mathrm{Re}(z + 2) = -1$
f) $|z - 5| = 6$
g) $|z + 2i| \geq 1$
h) $|z + i| = |z - i|$
i) $|z + 3| + |z + 1| = 4$
j) $|z + 3| - |z + 1| = \pm 1$

1.10 If z is a variable complex number, mark clearly on an Argand diagram (i.e., on the complex plane) the regions described by:

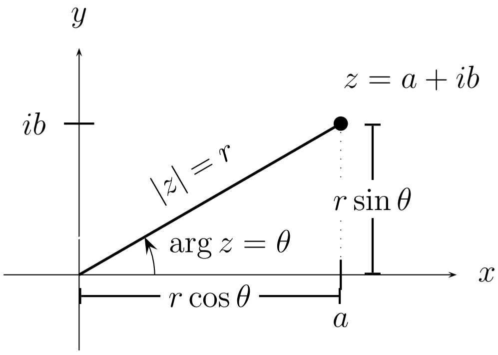
a) $\mathrm{Re}\,z \geq -2$ and $0 \leq \mathrm{Im}z \leq 3$
b) $\mathrm{Re}\,z \geq -2$ or $0 \leq \mathrm{Im}z \leq 3$
c) $2 < |z| < 3$ and $-\frac{\pi}{4} < \arg z < \frac{\pi}{4}$
d) $|z - 2 + i| > 1$ and $\mathrm{Re}\,z > 2$
e) $1 < |z - 2 + i| < 3$ and $-\frac{\pi}{3} < \arg z < \frac{\pi}{3}$.

CHAPTER 2

Polar form and roots of complex numbers

2.1 Polar and Cartesian forms of complex numbers

In the last chapter we introduced the set of complex numbers and showed how such numbers can be graphed on the complex plane. To graph any complex number $z = a + bi$ we need two pieces of information: the real part of the number a and the imaginary part of the number b. We can plot the same number, however, with two different pieces of information: the distance $|z|$ of the point from the origin and the angle of the line from the point to the origin, measured anti-clockwise from the positive real axis. This angle is known as the argument of z or arg$z$.



If we let $|z| = r$ and $\arg z = \theta$ then we can see that

$$\operatorname{Re} z = a = r\cos\theta \text{ and } \operatorname{Im} z = b = r\sin\theta.$$

Therefore, instead of writing $z = a + ib$, we can write

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

This is known as the polar form of a complex number. In polar form, a complex number is specified by its modulus r and its argument θ. The form of complex number $a+ib$, introduced in the last chapter, is called the Cartesian form.

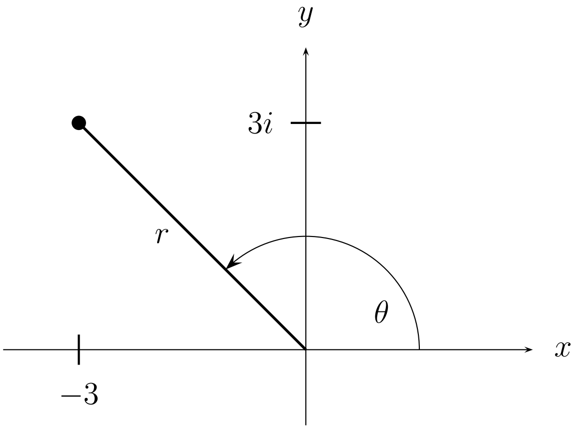
Complex numbers can easily be changed from one form to another. If a number is in Cartesian form $z = a + ib$, then the modulus $r = \sqrt{a^2 + b^2}$ and the argument θ can be found using

$\tan\theta = \frac{b}{a}$. Because $\tan\theta$ has the same values in the first and third quadrants and in the second and fourth quadrants *it is essential that you plot the complex number on the complex plane when you are finding its argument.* This will make it very clear in which quadrant the argument lies.

Examples 2.1a

i) Write $-3+3i$ in polar form.

Here $r = |-3+3i| = \sqrt{(-3)^2+3^2} = \sqrt{18} = 3\sqrt{2}$. Plotting $-3+3i$ on the complex plane gives:

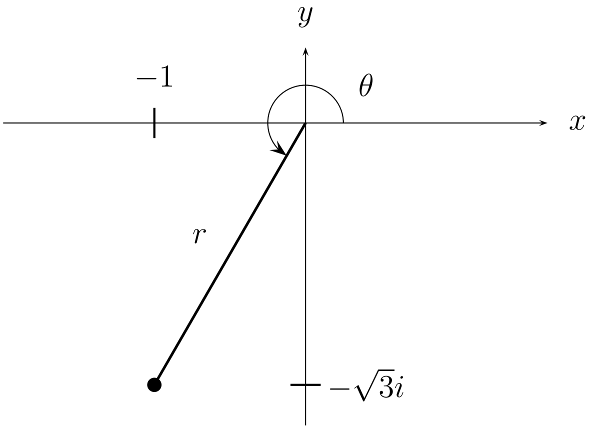


Hence $\arg(-3+3i)$ is in the second quadrant. By inspection we can see that $\arg(-3+3i) = 3\pi/4$. Alternatively we find that $\tan\theta = -1$ and hence $\theta = 3\pi/4$. Without the diagram we are left with the alternatives $\theta = -\pi/4$ or $3\pi/4$. As we know, $\tan^{-1}(-1) = -\frac{\pi}{4}$ (If you use a calculator, in radian mode, it will tell you that $\tan^{-1}(-1) \approx -0.7854$.) The diagram easily distinguishes between right and wrong answers. So

$$-3+3i = 3\sqrt{2}\left(\cos 3\pi/4 + i\sin 3\pi/4\right).$$

ii) Write $-1-\sqrt{3}i$ in polar form.

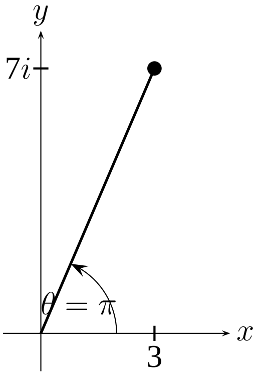
The modulus is given by $r = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$. Plotting $-1-\sqrt{3}i$ in the complex plane we have:



$\arg(-1+\sqrt{3}i)$ lies in the third quadrant. Since $\tan\theta = \sqrt{3}$ then $\theta = 4\pi/3$. (We could also write $\theta = -2\pi/3$ equally correctly.) Therefore $-1-\sqrt{3}i = 2(\cos 4\pi/3 + i\sin 4\pi/3)$ in polar form.

iii) Find the modulus and argument of $3+7i$.

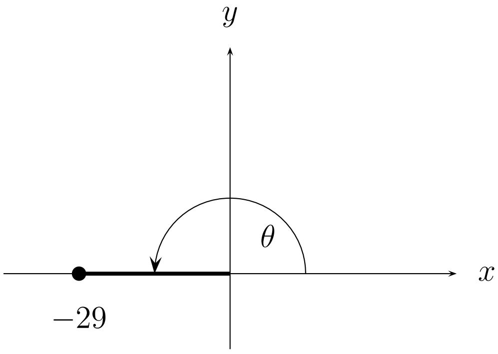
The modulus is $r = \sqrt{3^2+7^2} = \sqrt{58}$. In the complex plane $3+7i$ lies in the first quadrant:



We find that $\tan\theta = \frac{7}{3}$ and so $\theta = \tan^{-1}\frac{7}{3} \approx 1.17$. In polar form $3+7i = \sqrt{58}\left(\cos(\tan^{-1}\frac{7}{3}) + i\sin(\tan^{-1}\frac{7}{3})\right)$.

iv) Write -29 in polar form.

Although -29 is a real number it can still be written in polar form. Clearly $|-29| = 29$ and from the complex plane we see $\arg(-29) = \pi$.

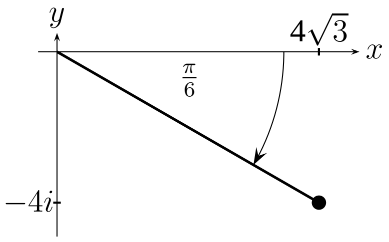


Hence $-29 = 29(\cos\pi + i\sin\pi)$ in polar form.

v) Convert $8(\cos(-\pi/6) + i\sin(-\pi/6))$ to Cartesian form.

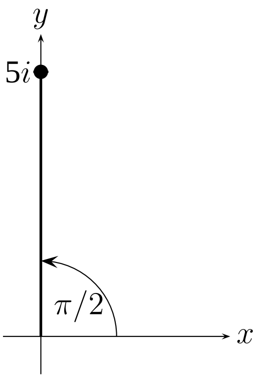
It is usually much simpler to convert a complex number from polar form to Cartesian form than to convert a complex number from Cartesian to polar form. All that needs to be done is to evaluate the cosine and sine and simplify the resulting expression. So

$$8(\cos(-\pi/6) + i\sin(-\pi/6)) = 8\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 4\sqrt{3} - 4i.$$



vi) Convert $5(\cos(\pi/2) + i\sin(\pi/2))$ into Cartesian form.

$$5(\cos(\pi/2) + i\sin(\pi/2)) = 5(0 + i) = 5i$$



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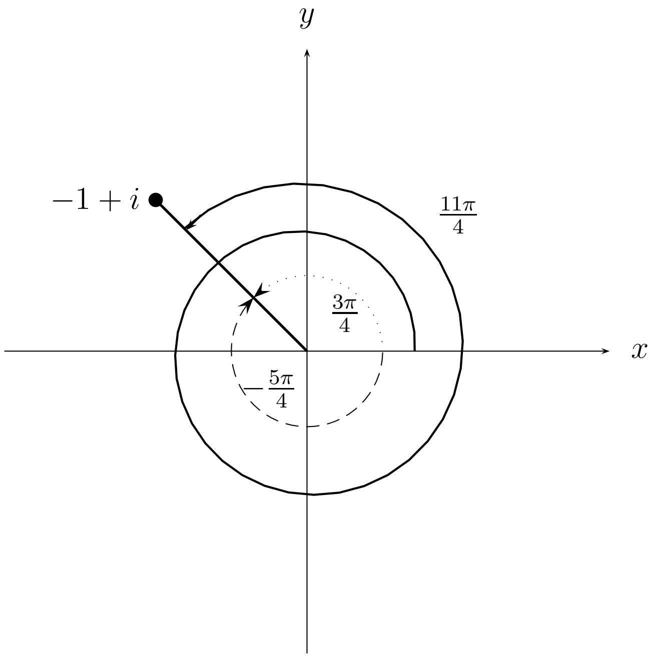
Sometimes the polar form of a complex number, $r(\cos\theta + i\sin\theta)$ is abbreviated to $r\operatorname{cis}\theta$ where

$$\operatorname{cis}\theta = \underline{c}\text{os}\theta + \underline{i}\,\underline{s}\text{in}\theta.$$

So, for example, $8\operatorname{cis}\left(\frac{-\pi}{6}\right) = 8\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$.

For a complex number z we can choose how to express $\arg z$. For example $\arg(-1+i)$ can be given as $3\pi/4$ or $-5\pi/4$ or $11\pi/4$. We could write it most generally as $3\pi/4 + 2k\pi$ where $k \in \mathbb{Z}$. In fact any complex number has an infinite number of arguments which all differ by integer multiples of 2π. This becomes important when we take roots of complex numbers later in this chapter.



To eliminate this ambiguity we can specify the principal argument of z, Arg$z$:

The principal argument of z, Arg$z$ is the particular argument of z such that

$$-\pi < \operatorname{Arg} z \leq \pi.$$

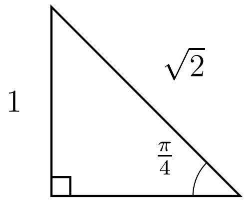
Hence $\arg(-1+i) = 3\pi/4$ or $-5\pi/4$ or $11\pi/4$ and so on, but Arg$z = 3\pi/4$ only.

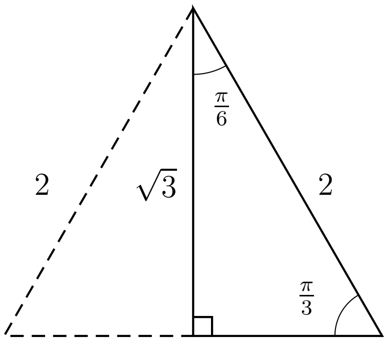
It is important to understand that complex numbers have multiple arguments when equating two complex numbers in polar form. Consider the complex numbers $z = r(\cos\theta + i\sin\theta)$

and $w = s(\cos\phi + i\sin\phi)$. If $z = w$ then they both correspond to the same point in the complex plane. Hence they are the same distance from the origin (that is, $r = s$) and have the same principal argument (that is, $\theta = \phi + 2k\pi$, where k is an integer).

▷ **Aside A note on special angles.** In the examples above you will see that most of the polar angles that we used were angles with exact sines or cosines, sometimes known as special angles. For example, any angle which is a multiple of $\pi/2$ has either sine or cosine equal to zero. So $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$ and $\cos\pi = -1$, $\sin\pi = 0$, for example.

The angles $\pi/6$ and $\pi/3$, which correspond to 30 and 60 degrees respectively, and any angels that are multiples of these have special values for sine and cosine, as does $\pi/4$ (45 degrees) and its multiples. You will have learnt about these special cases at high school. As they are used extensively in this chapter, it is important that you revise them as soon as possible if you have forgotten about them. You may find it helpful to look at the right-angle triangles with angle $\pi/4$ or $\pi/3$ and $\pi/6$.





It is easy to find sines and cosines from these triangles, since $\sin\theta$ is given by the length of the side opposite θ divided by the length of the hypotenuse and $\cos\theta$ is given by the length of the side adjacent to θ divided by the length of the hypotenuse. For example, $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$ and $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$. ◁

2.2 Arithmetic in polar form

Complex numbers in polar form can be added or multiplied together, subtracted one from the other and divided by one another or raised to a power. We shall see, however, that polar form is particularly useful when multiplying or dividing complex numbers of raising a complex number to a power.

Consider two complex numbers: $z = r(\cos\theta + i\sin\theta)$ and $w = t(\cos\phi + i\sin\phi)$. Here the modulus of z is r and the argument of z is θ. The modulus of w is t and its argument is ϕ.

Addition and subtraction

$$\begin{aligned} z + w &= r(\cos\theta + i\sin\theta) + t(\cos\phi + i\sin\phi) \\ &= r\cos\theta + ir\sin\theta + t\cos\phi + it\sin\phi \\ &= (r\cos\theta + t\cos\phi) + i(r\sin\theta + t\sin\phi). \end{aligned}$$

Subtraction is done in a similar way. Generally, there is little point in changing a complex number from Cartesian form to polar form to perform addition or subtraction. Using polar form for addition and subtraction is more complicated and gives no extra insight to the problem.

Example 2.2a

$$\begin{aligned} 6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)-2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) &= 6\cos\frac{\pi}{3}+6i\sin\frac{\pi}{3}-2\cos\frac{\pi}{6}-2i\sin\frac{\pi}{6} \\ &= \left(6\cos\frac{\pi}{3}-2\cos\frac{\pi}{6}\right)+i\left(6\sin\frac{\pi}{3}-2\sin\frac{\pi}{6}\right) \end{aligned}$$

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The solution in this example, although it involves sines and cosines is no longer in polar form; polar form is *strictly* in the form $r(\cos\theta+i\sin\theta)$

Multiplication

$$\begin{aligned} zw &= r(\cos\theta+i\sin\theta)t(\cos\phi+i\sin\phi) \\ &= rt(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi) \\ &= rt(\cos\theta\cos\phi+i\cos\theta\sin\phi+i\sin\theta\cos\phi+i^2\sin\theta\sin\phi) \\ &= rt((\cos\theta\cos\phi-\sin\theta\sin\phi)+i(\cos\theta\sin\phi+\sin\theta\cos\phi)) \\ &= rt(\cos(\theta+\phi)+i\sin(\theta+\phi)). \end{aligned}$$

The last line uses the angle sum formulae of trigonometry:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta. \end{aligned}$$

We see that the modulus of the product is rt, the product of the two moduli of the numbers which we multiplied together and that the argument of the product is $(\theta+\phi)$, the sum of the arguments of the original two numbers. In general

To **multiply** complex numbers in polar form *multiply* the moduli and *add* the arguments. That is,

$$(r(\cos\theta+i\sin\theta))\,(t(\cos\phi+i\sin\phi))=rt\,(\cos(\theta+\phi)+i\sin(\theta+\phi))\,.$$

Example 2.2b

$$\left(6(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})\right)\left(2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})\right) = 12(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$$

We can write down the answer straight away as the modulus will be the product of 6 and 2 and the argument will be $\pi/3+\pi/6=\pi/2$. ◊

Division

$$\begin{aligned}\frac{z}{w} &= \frac{r(\cos\theta + i\sin\theta)}{t(\cos\phi + i\sin\phi)}\\ &= \frac{r(\cos\theta + i\sin\theta)t(\cos\phi - i\sin\phi)}{t(\cos\phi + i\sin\phi)t(\cos\phi - i\sin\phi)}\end{aligned}$$

You should note that a complex number in the form $t(\cos\phi - i\sin\phi)$ is *not* in polar form and so the rule for multiplication in polar form does not apply. However, a number of the form $t(\cos\phi - i\sin\phi)$ can easily be put into polar form because

$$-\sin\phi = \sin(-\phi) \quad \text{and} \quad \cos\phi = \cos(-\phi).$$

Consequently,

$$t(\cos\phi - i\sin\phi) = t(\cos(-\phi) + i\sin(-\phi)),$$

so we have written $\frac{z}{w}$ in polar form. Hence, we have

$$\begin{aligned}\frac{z}{w} &= \frac{r(\cos\theta + i\sin\theta)t(\cos(-\phi) + i\sin(-\phi))}{t(\cos\phi + i\sin\phi)t(\cos\phi - i\sin\phi)}\\ &= \frac{rt(\cos\theta + i\sin\theta)(\cos(-\phi) + i\sin(-\phi))}{t^2(\cos^2\phi + \sin^2\phi)}\\ &= \frac{r}{t}(\cos(\theta - \phi) + i\sin(\theta - \phi)).\end{aligned}$$

We see that the modulus of the quotient is r/t, the quotient of the two moduli of the original two numbers and that the argument of the quotient is $\theta - \phi$, the difference of the arguments of the original two numbers. In general

> To divide complex numbers in polar form we *divide* their moduli and *subtract* their arguments.

Example 2.2c

$$\frac{6(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})}{2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})} = \frac{6}{2}\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right) \quad = \quad 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

We can write down the answer immediately as the modulus will be the quotient of 6 and 2 and the argument will be $\pi/3 - \pi/6 = \pi/6$. ♢

Raising to an integer power Let us consider the problem $(r(\cos\theta + i\sin\theta))^2$. Using the rule for multiplication in polar form this becomes $r^2(\cos 2\theta + i\sin 2\theta)$. Following on from this we can write

$$(r(\cos\theta + i\sin\theta))^3 = r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta).$$

Similarly $(r(\cos\theta+i\sin\theta))^4 = r^4(\cos 4\theta+i\sin 4\theta)$. It is easy to see that for $n \in \{1,2,3,\ldots\}$

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta).$$

✍ It is a useful exercise to prove this by induction. In fact, this is true not only when n is a positive integer but for all integer values of n.

> To raise a complex number to any integer, raise the modulus to the integer and multiply the argument by the integer.

Example 2.2d

$$\begin{aligned}\left(6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^8 &= 6^8\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)\\ &= 6^8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).\end{aligned}$$

since $\cos 8\pi/3 = \cos 2\pi/3$ and $\sin 8\pi/3 = \sin 2\pi/3$. ◊

In the special case when a complex number of modulus 1 is raised to an integer power, we have De Moivre’s theorem .

> For any $n \in \mathbb{Z}$,
>
> $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Examples 2.2e

i)

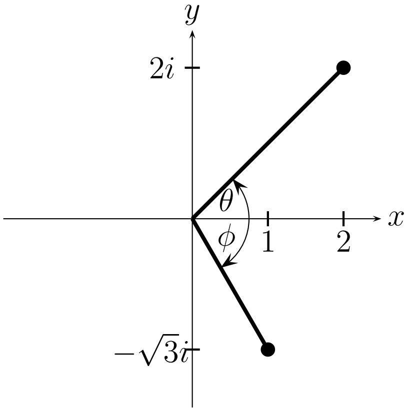
$$3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(4(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2})\right) = 12\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

ii)

$$\frac{3(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})}{4(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2})} = \frac{3}{4}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}).$$

iii) Find $(2+2i)(1-\sqrt{3}i)$ in polar form.

First, let us put both numbers into polar form. This simplifies the multiplication and we will also need these numbers in polar form for the next example. It is essential to draw a diagram:



Here $|2+2i| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ and $|1-\sqrt{3}i| = \sqrt{1+3} = 2$. From the diagram, $\theta = \arg(2+2i)$ is in the first quadrant and $\phi = \arg(1-\sqrt{3}i)$ is in the fourth quadrant. Since $\tan\theta = 1$, $\theta = \frac{\pi}{4}$ and since $\tan\phi = \sqrt{3}$, $\phi = -\pi/3$. So we have

$$\begin{aligned}(2+2i)(1-\sqrt{3}i) &= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) 2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)\\ &= 4\sqrt{2}\left(\cos(\frac{\pi}{4} + \frac{-\pi}{3}) + i\sin(\frac{\pi}{4} + \frac{-\pi}{3})\right)\\ &= 4\sqrt{2}\left(\cos(\frac{-\pi}{12}) + i\sin(\frac{-\pi}{12})\right).\end{aligned}$$

iv) Find $(2+2i)/(1-\sqrt{3}i)$ in polar form.

The numbers $(2+2i)$ and $(1-\sqrt{3}i)$ are already in polar form from the previous example.

$$\begin{aligned}\frac{(2+2i)}{(1-\sqrt{3}i)} &= \frac{2\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})}{2(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3})}\\ &= \sqrt{2}\left(\cos(\frac{\pi}{4} - \frac{-\pi}{3}) + i\sin(\frac{\pi}{4} - \frac{-\pi}{3})\right)\\ &= \sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right).\end{aligned}$$

v) Find $\left((2+2i)/(1-\sqrt{3}i)\right)^6$.

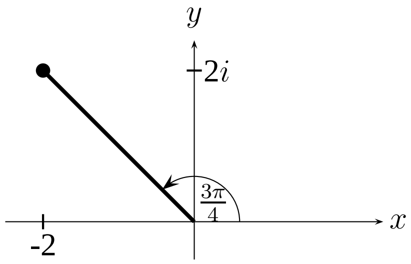
The quotient has already been calculated in polar form in the previous example.

$$\begin{aligned}\left(\frac{(2+2i)}{(1-\sqrt{3}i)}\right)^6 &= \left(\sqrt{2}(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})\right)^6\\ &= (2^{\frac{1}{2}})^6\left(\cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}\right)\\ &= 2^3\left(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}\right)\\ &= -8i.\end{aligned}$$

◊

2.3 Roots of complex numbers

What is meant by “a root of a complex number”, and how could such numbers be found? For example, what is a cube root of $-2+2i$? By analogy with roots of real numbers, an obvious answer is to say it’s a complex number z whose cube is $-2+2i$. It turns out that the easiest way to find z is to use polar form; if we let $z = r(\cos\theta + i\sin\theta)$ then our task is to find all values of r and all values of θ such that $z^3 = r^3(\cos 3\theta + i\sin 3\theta) = -2+2i$. Let’s also put $-2+2i$ into polar form:



The modulus is $|-2+2i| = \sqrt{8}$. The diagram shows that the principal argument is $3\pi/4$ (in the second quadrant) and so $-2+2i = \sqrt{8}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$. Then we have

$$r^3(\cos 3\theta + i\sin 3\theta) = \sqrt{8}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}).$$

Since the left and right hand sides are both represented by the same point in the complex plane, we must have $r^3 = \sqrt{8}$ and $3\theta = \frac{3\pi}{4} + 2k\pi$ where k can be any integer. Therefore

$$r = \sqrt{2} \quad \text{and} \quad \theta = \pi/4 + 2k\pi/3.$$

The unknown z we seek is then given in its most general form as

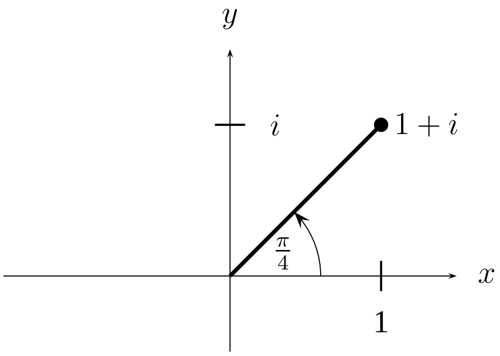
$$z = \sqrt{2}(\cos(\pi/4 + 2k\pi/3) + i\sin(\pi/4 + 2k\pi/3)).$$

Since k can be any integer it appears at first sight that there are infinitely many complex numbers z whose cube is $-2+2i$! In fact, it turns out that there are exactly three. To see this, let’s experiment with some different values of the integer k.

When $k=0$, we obtain

$$\begin{aligned} z &= \sqrt{2}(\cos\pi/4 + i\sin\pi/4) \\ &= \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) \\ &= 1 + i. \end{aligned}$$

Plotting this answer on the complex plane we get:



When $k = 1$, we obtain

$$\begin{aligned} z &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)\right) \\ &= \sqrt{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right) \end{aligned}$$

When $k = 2$, we obtain

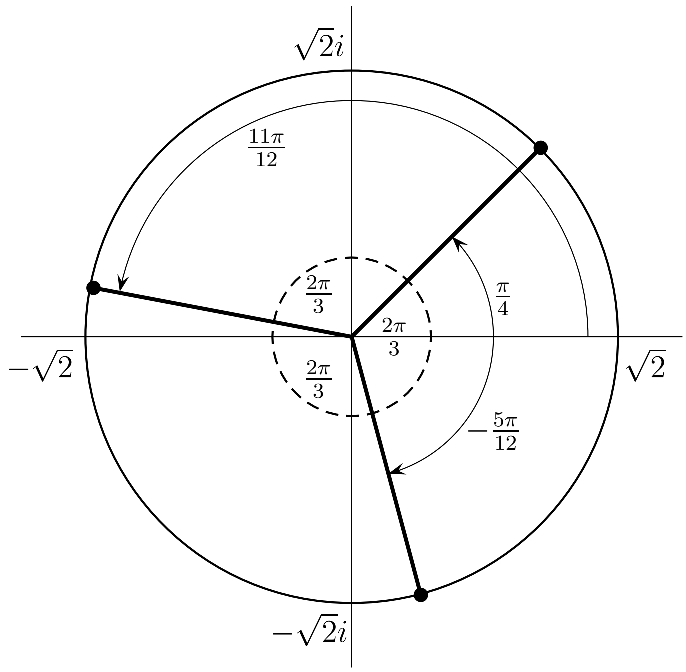
$$\begin{aligned} z &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)\right) \\ &= \sqrt{2}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right) \\ &= \sqrt{2}\left(\cos\frac{-5\pi}{12}+i\sin\frac{-5\pi}{12}\right) \end{aligned}$$

If we choose other values of k it turns out that we simply replicate one of the three values of z that we've already calculated. For example, if $k = -1$, then

$$\begin{aligned} z &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{-2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{-2\pi}{3}\right)\right) \\ &= \sqrt{2}\left(\cos\left(\frac{-5\pi}{12}\right)+i\sin\left(\frac{-5\pi}{12}\right)\right), \end{aligned}$$

which is one of the values already found.

If all three distinct solutions are plotted on the complex plane we see that all lie on the circle, of radius $\sqrt{2}$, centred on the origin and each is separated from the others by an angle of $2\pi/3$. The complex number $-2+2i$ has exactly three distinct cube roots. The reason for this will become clear in the next section.



How many complex roots does a real number have? Let us look at the fourth roots of 16. You already know that $2^4 = (-2)^4 = 16$. Hence 2 and -2 are fourth roots of 16. Are there other fourth roots?

We are looking for all z such that $z^4 = 16$. Writing $z = r(\cos\theta + i\sin\theta)$ and $16 = 16(\cos 0 + i\sin 0)$ we have

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)$$

and hence $r^4 = 16$ and $4\theta = 0 + 2k\pi = 2k\pi$, where k can be any integer. Therefore $r = 2$ and $\theta = \frac{k\pi}{2}$, for any integer k. This gives $z = 2(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2})$.

We shall now choose various values of k to find explicit values of z.

When $k = 0$ we obtain

$$z = 2(\cos 0 + i\sin 0) = 2.$$

When $k = 1$ we obtain

$$z = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i.$$

When $k = 2$ we obtain

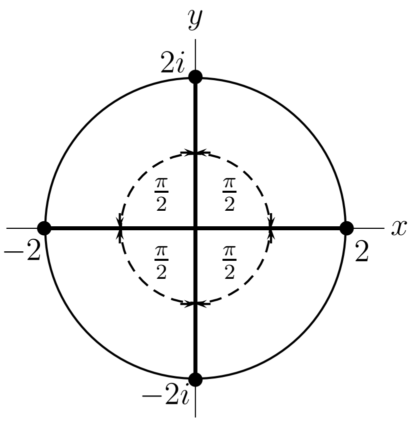
$$z = 2(\cos\pi + i\sin\pi) = -2,$$

and when $k = 3$ we obtain

$$z = \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i.$$

All other values of k give one of the four answers already found, namely ± 2, $\pm 2i$. For example, if $k = 7$, we obtain $z = 2(\cos\frac{7\pi}{2} + i\sin\frac{7\pi}{2}) = -2i$.

Therefore, 16 has two *real* fourth roots but it has four *complex* fourth roots. When these roots are plotted on the complex plane, they all lie on the circle radius 2, spaced at angle $\pi/2$ apart.



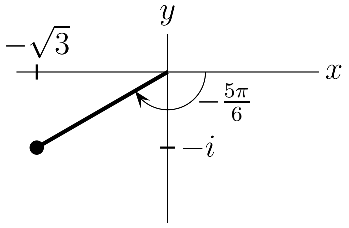
In fact, every non–zero complex number (which includes every real number) has two complex square roots, three complex cube roots, four complex fourth roots and so on. In general

Every non–zero complex number has n distinct complex nth roots.

Therefore if we seek to find all cube roots of a complex number, for example, we know that there will be three cube roots. Knowing how many roots to look for is useful in deciding how many different values of k to use in finding roots.

Example 2.3a Find all fifth roots of $-\sqrt{3}-i$, that is, all z such that $z^5 = -\sqrt{3}-i$.

First, we put $-\sqrt{3}-i$ into polar form. The modulus of $-\sqrt{3}-i$ is $|-\sqrt{3}-i| = 2$. Plotting $-\sqrt{3}-i$ on the complex plane we see that the principal argument of $-\sqrt{3}-i$ is $-\frac{5\pi}{6}$.



Writing $z = r(\cos\theta + i\sin\theta)$ then gives

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta) = 2\left(\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right).$$

Therefore $r = 2^{\frac{1}{5}}$ and $\theta = \frac{-\pi}{6} + \frac{2k\pi}{5}$, for any integer k. The required values of z are then obtained by setting $k = 0,\ 1,\ 2,\ 3$ and 4 in the equation

$$z = 2^{\frac{1}{5}}\left(\cos\left(\frac{-\pi}{6} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{-\pi}{6} + \frac{2k\pi}{5}\right)\right).$$

We obtain five different values of z:

$$z = 2^{\frac{1}{5}}\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right),$$

$$z = 2^{\frac{1}{5}}\left(\cos\left(\frac{7\pi}{30}\right) + i\sin\left(\frac{7\pi}{30}\right)\right),$$

$$z = 2^{\frac{1}{5}}\left(\cos\left(\frac{19\pi}{30}\right) + i\sin\left(\frac{19\pi}{30}\right)\right),$$

$$z = 2^{\frac{1}{5}}\left(\cos\left(\frac{31\pi}{30}\right) + i\sin\left(\frac{31\pi}{30}\right)\right),$$

$$z = 2^{\frac{1}{5}}\left(\cos\left(\frac{43\pi}{30}\right) + i\sin\left(\frac{43\pi}{30}\right)\right).$$

These are the five fifth roots of $-\sqrt{3} - i$.

◊

2.4 Roots of polynomial equations

We have already seen how to find roots of a complex number by understanding that the roots are solutions of a very simple type of polynomial equation. What can be said about solutions of more complicated polynomial equations such as $z^4 - 18z^2 + 192z - 175 = 0$? To discuss this more general type of equation we need to be clear on what we mean by a polynomial and how we can use different number sets in writing down and solving polynomial equations.

A **polynomial** in z is an expression of the form

$$a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

where z is known as the **variable** and the numbers $a_n, a_{n-1}, \ldots, a_1$ and a_0 are **coefficients.**

If $a_n \neq 0$ then the polynomial is said to have **degree** n. The term $a_n z^n$ is known as the **leading term.**

The **roots of a polynomial equation** are the numbers z which satisfy

$$a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 = 0.$$

It is important to know in which number set the solutions of a polynomial equation lie. For example, the problem “solve $z^4 - 16 = 0$ over the real numbers” (or equivalently, “find the real roots of $z^4 - 16 = 0$”) has the answer $z = 2$ or -2. If the problem is changed slightly to read "solve $z^4 - 16 = 0$ over the complex numbers" (or "find the complex roots of $z^4 - 16 = 0$"), then the correct answer is $z = 2,\, 2i, -2$ or $-2i$. Clearly this polynomial equation has more complex solutions than real solutions. It is a fourth order equation and has four complex roots, although it has only two real roots.

In general

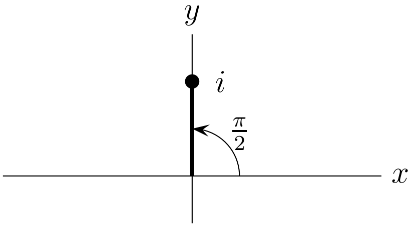
A polynomial equation of degree n has at most n complex roots. All, some or none of these roots may be real.

Example 2.4a Find all complex roots of the polynomial equation $z^5 - iz^2 = 0$.

Observe that the left hand side can be factorised; this gives

$$z^2(z^3 - i) = 0.$$

Thus the roots of the original equation consist of all the roots of the equation $z^2 = 0$ together with all the roots of the equation $z^3 - i = 0$. The only root of $z^2 = 0$ is $z = 0$. We now find the roots of $z^3 - i = 0$. We write i in polar form.



Clearly $|i| = 1$ and $\arg i = \pi/2 + 2k\pi$. As we are seeking cube roots, we expect three solutions, so we will use $k = 0, 1, 2$ in the expression

$$1^{\frac{1}{3}}(\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right)).$$

When $k = 0$ we obtain $\cos\pi/6 + i\sin\pi/6$, when $k = 1$ we obtain $\cos 5\pi/6 + i\sin 5\pi/6$ and when $k = 2$ we obtain $\cos 3\pi/2 + i\sin 3\pi/2$. Therefore the complex roots of $z^3 - i = 0$ are $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and $-i$.

There are just four roots of the original equation, namely

$$0, \quad \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad \frac{-\sqrt{3}}{2} + \frac{1}{2}i, \quad -i.$$

◊

Technical aside Most polynomial equations of degree n will have *exactly* n complex roots. The reason that we have to be careful here is that some equations have repeated roots. For example we say the polynomial $z^2 - 2z + 1 = (z-1)^2$ has a *double root* at $z = 1$, also called a root of *multiplicity* 2. If n is a positive integer then the equation $z^n = 0$ has a root of multiplicity n at $z = 0$. It is true that every polynomial equation of degree n has *exactly* n complex roots, *counted with multiplicity*. ◁

We have already seen in Chapter 1 that when a quadratic equation with real coefficients has non–real complex roots, then these roots come in complex conjugate pairs. So for example, the equation $z^2 + 4z + 5 = 0$ has roots $z = -2 + i$ and $z = -2 - i$.

In fact

> If the coefficients of a polynomial equation are *all real* then all of the non–real complex roots occur in complex conjugate pairs.

For example, the polynomial equation $z^4 - 16 = 0$ that was discussed earlier in this chapter, has two real roots 2 and -2 and two imaginary roots $2i$ and $-2i$ and these imaginary roots are complex conjugates of each other. The coefficients of this polynomial, 1 and -16 are both real and so we expect complex roots will occur in complex conjugate pairs. By contrast, the polynomial $z^3 - i = 0$, solved in Example 2.4a above, does not have all real coefficients; the coefficients are 1, which is real and $-i$ which is not real. The roots of $z^3 - i = 0$ are $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and $-i$. Although they are all non–real complex numbers, they do not occur in complex conjugate pairs.

If one complex root of a polynomial equation with real coefficients is known then its complex conjugate can immediately be written down to give another root.

Example 2.4b Find all roots of $z^4 - 18z^2 + 192z - 175 = 0$, given that $3 - 4i$ is a root.

If $3 - 4i$ is a root, then its complex conjugate $3 + 4i$ is also a root. We write down the quadratic expression with these roots. We can then divide this into the original polynomial to get another quadratic which can be easily solved. So $(z-(3-4i))(z-(3+4i)) = z^2-6z+25$ and we want to find $(z^4-18z^2+192z-175)/(z^2-6z+25)$. Using polynomial long division:

$$\begin{array}{rrrrrr}
 & & & z^2 & +6z & -7 \\ \cline{2-6}
z^2-6z+25 & \big) \, z^4 & & -18z^2 & +192z & -175 \\
 & z^4 & -6z^3 & +25z^2 & & \\ \cline{2-6}
 & & 6z^3 & -43z^2 & +192z & \\
 & & 6z^3 & -36z^2 & +150z & \\ \cline{3-6}
 & & & -7z^2 & +42z & -175 \\
 & & & -7z^2 & +42z & -175 \\ \cline{4-6}
 & & & & & 0
\end{array}$$

Therefore we have

$$\begin{aligned} z^4 - 18z^2 + 192z - 175 &= (z-(3-4i))(z-(3+4i))(z^2+6z-7) \\ &= (z-(3-4i))(z-(3+4i))(z-1)(z+7). \end{aligned}$$

So the four roots of the polynomial are $3-4i$, $3+4i$, 1 and -7. $\Diamond$

Technical aside It is not difficult to prove rigorously that if a polynomial equation with real coefficients has complex roots then these roots occur in complex conjugate pairs.

✍ First we need to show that $\overline{z+w} = \overline{z} + \overline{w}$ and that $\overline{zw} = \overline{z}\,\overline{w}$.Try doing this by writing $z = a+ib$ and $w = c+id$ and calculating $\overline{z}+\overline{w}$ and $\overline{z}\,\overline{w}$.

Then, let us consider a polynomial

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

where a_n, a_{n-1}, a_{n-2}, $\ldots$, a_0 are all real.

Suppose there is some complex number v which is a root of $p(v) = 0$. We want to show that $p(\overline{v}) = 0$ also. If we take complex conjugates of both sides of the equation we have $\overline{p(v)} = \overline{0} = 0$ and hence

$$\begin{aligned} 0 &= \overline{p(v)} \\ &= \overline{a_n v^n + a_{n-1}v^{n-1} + a_{n-2}v^{n-2} + \cdots + a_1 v + a_0} \\ &= \overline{a_n v^n} + \overline{a_{n-1}v^{n-1}} + \overline{a_{n-2}v^{n-2}} + \cdots + \overline{a_1 v} + \overline{a_0} \\ &= \overline{a_n}\,\overline{v^n} + \overline{a_{n-1}}\,\overline{v^{n-1}} + \overline{a_{n-2}}\,\overline{v^{n-2}} + \cdots + \overline{a_1}\,\overline{v} + a_0 \\ &= a_n\overline{v^n} + a_{n-1}\overline{v^{n-1}} + a_{n-2}\overline{v^{n-2}} + \cdots + a_1\overline{v} + a_0 \\ &= a_n(\overline{v})^n + a_{n-1}(\overline{v})^{n-1} + a_{n-2}(\overline{v})^{n-2} + \cdots + a_1(\overline{v}) + a_0 \\ &= p(\overline{v}) \end{aligned}$$

✍ As you read this proof try to work out why each line follows from the previous line. ◁

Exercises

In addition to doing the following exercises you should look at the on–line quiz

`www.maths.usyd.edu.au/u/UG/JM/MATH1001/Quizzes/quiz2.html`

which covers the material in this chapter. You can get to this page from the course homepage.

2.1 For each of the following numbers, give the numerical value of the real part x, the imaginary part y, the modulus r and the principal value of the argument θ. Plot the number as a point in the complex plane.

a) $1 - i\sqrt{3}$

b) $1/(1-i)$

c) $(i+\sqrt{3})^2$

d) $2(\cos(\pi/6) + i\sin(\pi/6))$

e) $\left(\dfrac{1+i}{1-i}\right)^2$

f) $\dfrac{3+i}{2+i}$.

2.2 Write each of the following complex numbers in polar form:

$$-4i \qquad -2+2i \qquad 1-i$$

Use these results to perform the following operations in polar form:

a) $(-2+2i)(1-i)$

b) $-4i/(-2+2i)$

c) $(1-i)^6$

d) $(-2+2i)^{15}$.

2.3 Use de Moivre's theorem to simplify:

a) $(\cos(2\pi/3) + i\sin(2\pi/3))^9$

b) $(\cos(\pi/3) + i\sin(\pi/3))^4$

c) $(\cos(2\pi/3) - i\sin(2\pi/3))^6$

d) $(\sin(2\pi/3) + i\cos(2\pi/3))^9$.

2.4 Recall that if $p(z)$ is a polynomial with real coefficients and if $z \in \mathbb{C}$ is a root of $p(z)$ then so is $\overline{w}$. Find the roots of the quadratic equation $q(z) = z^2 - 3(1+i)z - 2 + 6i = 0$. Verify that if w is a root of $q(z)$ then $\overline{w}$ is not a root and explain why this does not contradict the statement at the start of this question.

2.5 Find all the roots of $f(z) = z^4 - 3z^3 + 7z^2 + 21z - 26$, given that $2-3i$ is a root.

2.6 Find all the roots of $z^4 - 5z^3 + 4z^2 + 2z - 8$, given that $1-i$ is a root.