Answers to Extended Response, SS1002 2011

1. (a) $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$,
   $\overrightarrow{AC} = 3\mathbf{j} - \mathbf{k}$.

   (b) $\overrightarrow{AB} \times \overrightarrow{AC} = 10\mathbf{i} + \mathbf{j} + 3\mathbf{k}$,
   so the equation for the plane is $10x + y + 3z = 10(1) + (-2) + 3(3) = 17$.

   (c) We are looking for the equation of the line through A and B.
   This is given by the vector equation

$$\mathcal{L}: \quad \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}), \quad t \in \mathbb{R},$$

   which has Cartesian equations

$$x - 1 = \frac{y+2}{2} = \frac{z-3}{-4}. \quad \text{(1 mark).}$$

   Alternatively, students may compute $\mathbf{n}_1 \times \mathbf{n}_2 = (10\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \times (4\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) = -9(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$ to arrive at the same (up to scaling) Cartesian equations.

2. (a) Suppose $\gamma\mathbf{u} + \delta\mathbf{v} = 0$. Applying M to both sides gives $-\gamma\mathbf{u} + \delta\mathbf{v} = 0$.
   Adding these equations together forces $\delta = 0$, and hence $\gamma = 0$.

   (b)(i) The eigenvalues of M are 1 and -1 .

   (b)(ii) The eigenvalues of M^2 are all 1 .

   (c) Since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent and have nonzero eigenvalues ,
   the determinant of M must be nonzero, hence it must be invertible .

   (d) We have the following matrix equation:

$$M \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

   and hence

$$M = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}^{-1},$$

   where

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix},$$

   and therefore

$$M = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

3. (a) $\overrightarrow{AB} = \mathbf{a} - \mathbf{b}$, $\overrightarrow{AC} = \mathbf{a} - \mathbf{c}$, $\overrightarrow{BC} = \mathbf{b} - \mathbf{c}$.

   (b) We evaluate

$$\begin{aligned}(\mathbf{b}+\mathbf{c}) \cdot (\mathbf{b}-\mathbf{c}) &= \mathbf{b}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{b} - \mathbf{c}\cdot\mathbf{c} \\ &= |\mathbf{b}|^2 - |\mathbf{c}|^2 \\ &= 0,\end{aligned}$$

   as $|\mathbf{b}| = |\mathbf{c}|$, hence the vectors are orthogonal .

(c) The line through A and H is perpendicular to $\overrightarrow{BC} = \mathbf{b} - \mathbf{c}$, as it is an altitude. Hence it must be parallel to $\mathbf{b} + \mathbf{c}$, and contains the point A .

(d) The lines intersect when

$$\mathbf{a} + t(\mathbf{b} + \mathbf{c}) = \mathbf{c} + s(\mathbf{a} + \mathbf{b}),$$

and taking the dot product of both sides with $\mathbf{b} - \mathbf{c}$ gives

$$\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{c} + s\mathbf{a} \cdot \mathbf{b} - s\mathbf{a} \cdot \mathbf{c} + s\mathbf{b} \cdot \mathbf{b} - s\mathbf{b} \cdot \mathbf{c}.$$

Rearranging, we have

$$\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} + |\mathbf{c}|^2 = s(\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} + |\mathbf{b}|^2),$$

and since $|\mathbf{b}| = |\mathbf{c}|$, we must have $s = 1$. The argument that $\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} + |\mathbf{c}|^2$ is nonzero is not required.
We therefore have shown that $\mathcal{L}_1$ and $\mathcal{L}_2$ meet at the point whose position vector is $\mathbf{a} + \mathbf{b} + \mathbf{c}$, which must also be the position vector of the point H.