THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Summer School	**Exercise Sheet 1**	2012

There are six teaching weeks. There will usually be two sets of exercises each week. This is the first set of exercises. Preparatory exercises ideally should be attempted before coming to the tutorial and solutions are provided. All exercises may be attempted and students work at their own pace, individually or in groups. As a guide, a suggestion is given for exercises to be completed during and after the tutorial. Occasionally an exercise is indicated as suitable for group work or discussion. Questions labelled with an asterisk are suitable for students aiming for a credit or higher. Thursday tutorials for the first four weeks will include two *(Homework)* questions, to be completed and handed to your tutor in your Friday tutorial.

Important Ideas and Useful Facts:

(i) A *geometric vector* $\mathbf{v}$ is a directed line segment in space, described by its length $|\mathbf{v}|$ and direction. Two vectors are *equal* if they have the same magnitude and direction, regardless of their position in space.

(ii) A *scalar* λ is a real number. The *scalar multiple* $\lambda\,\mathbf{v}$ has length $|\lambda||\mathbf{v}|$ and the same direction as $\mathbf{v}$ if λ is positive, and opposite direction if λ is negative.

(iii) If P and Q are points in space then $\overrightarrow{PQ}$ denotes the vector pointing from P to Q. The *position vector* of the point P is the vector $\overrightarrow{OP}$ where O denotes the origin in space.

(iv) A *parallelogram* is a quadrilateral such that two opposite sides are parallel and have the same length (which implies that the other two opposite sides are also parallel and have the same length).

(v) Parallelogram Law of Vector Addition: The *vector sum* $\mathbf{v}+\mathbf{w}$ is represented by the diagonal of the parallelogram formed using sides $\mathbf{v}$ and $\mathbf{w}$.

(vi) Commutative Law of Addition: $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

(vii) Associative Law of Addition: $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.

(viii) The zero vector $\mathbf{0}$ has zero length and points in every direction. For every vector $\mathbf{v}$, $\mathbf{0}+\mathbf{v}=\mathbf{v}$ and $0\mathbf{v}=\mathbf{0}$.

(ix) The *negative* of $\mathbf{v}$ is $-\mathbf{v}=(-1)\mathbf{v}$ with the same length as $\mathbf{v}$, but pointing in the opposite direction. If P and Q are points then $\overrightarrow{QP}=-\overrightarrow{PQ}$.

(x) The *vector difference* $\mathbf{v}-\mathbf{w}$ equals $\mathbf{v}+(-\mathbf{w})$ and has the property that $\mathbf{w}+(\mathbf{v}-\mathbf{w})=\mathbf{v}$.

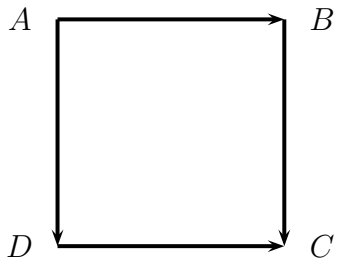
(xi) If $\mathbf{v}$ and $\mathbf{w}$ are vectors and λ and μ are scalars, then

$$\lambda(\mu\mathbf{v})=(\lambda\mu)\mathbf{v}\,,\quad \lambda(\mathbf{v}+\mathbf{w})=\lambda\mathbf{v}+\lambda\mathbf{w}\,,\quad (\lambda+\mu)\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}\,,$$

$$-(-\mathbf{v})=\mathbf{v}\,,\quad \mathbf{v}-\mathbf{v}=\mathbf{0}\,,\quad 1\mathbf{v}=\mathbf{v}\,,\quad (-\lambda)\mathbf{v}=-(\lambda\mathbf{v})\,.$$

Preparatory Exercises (answers below):

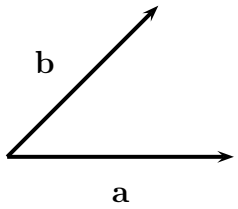
1. The edges of the square $ABCD$ are marked by vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$, $\overrightarrow{DC}$, as shown.

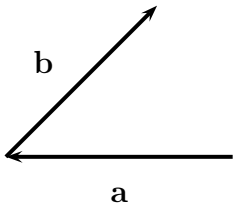


True or false:

(i) $\overrightarrow{AB} = \overrightarrow{BC}$ (ii) $\overrightarrow{AB} = \overrightarrow{CD}$ (iii) $\overrightarrow{AD} = \overrightarrow{BC}$ (iv) $\overrightarrow{AC} = \overrightarrow{BC} + \overrightarrow{DC}$

2. Draw the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ on each diagram.





3. Simplify the following vector expressions.

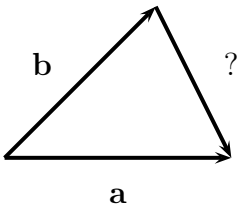
(i) $3\mathbf{a} + 2\mathbf{b} - 4(\mathbf{b} + \frac{1}{2}\mathbf{a})$ (ii) $-(\mathbf{w} - 6\mathbf{z}) - 2\mathbf{w} + \mathbf{v} - 2\mathbf{z}$

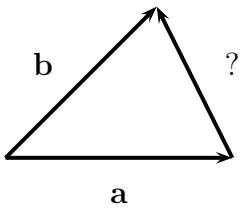
Tutorial Exercises:

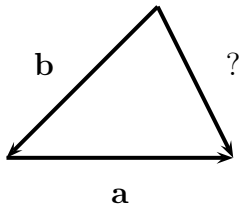
4. If $|\mathbf{v}| = 2$, find $|\mathbf{u}|$ in each of the following cases.

(i) $\mathbf{u} = 3\mathbf{v}$ (ii) $\mathbf{u} = \frac{1}{2}\mathbf{v}$ (iii) $\mathbf{u} = -3\mathbf{v}$ (iv) $\mathbf{v} = 3\mathbf{u}$

5. In each diagram below, find the unknown vector in terms of $\mathbf{a}$ and $\mathbf{b}$.







6. Solve for $\mathbf{x}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in each case.

(i) $\mathbf{v}+\mathbf{x} = \mathbf{u}-\mathbf{w}$ (ii) $\mathbf{v}-\mathbf{x} = \mathbf{w}-\mathbf{u}$ (iii) $2\mathbf{v}+\mathbf{x} = 2\mathbf{w}-2\mathbf{u}-\mathbf{x}$

7. A balloon experiences two forces, a buoyancy force of 8 newtons vertically upwards and a wind force of 6 newtons acting horizontally to the right. Calculate the magnitude and direction of the resultant force.

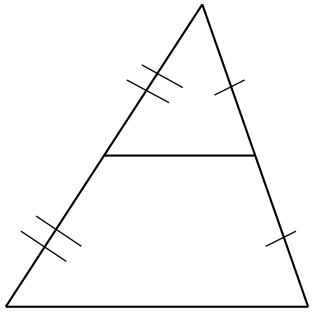
8. (suitable for group discussion) Explain the associative law for addition of vectors.

9. (suitable for group discussion) Explain, geometrically, the *triangle inequality*

$$|\mathbf{v}+\mathbf{w}| \leq |\mathbf{v}|+|\mathbf{w}| \,,$$

and determine when equality occurs.

10.* Prove, using vectors, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side.



Further Exercises:

11. Express $2\mathbf{a}-3\mathbf{b}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, and simplify, when

$$\mathbf{a} = \mathbf{u}+\mathbf{v} \,, \qquad \mathbf{b} = 3\mathbf{u}-2\mathbf{v} \,.$$

12. Let $ABCDEF$ be a regular hexagon and put

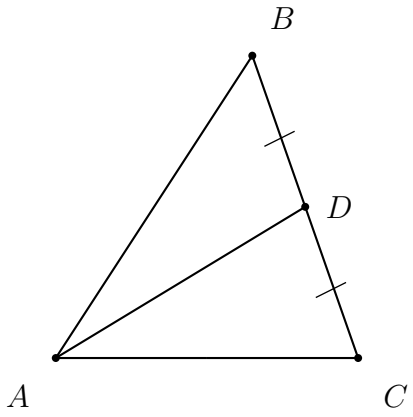
$$\mathbf{a} = \overrightarrow{AB} \,, \qquad \mathbf{b} = \overrightarrow{BC} \,.$$

Find vector expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the displacements

$$\overrightarrow{CD}, \quad \overrightarrow{DE}, \quad \overrightarrow{EF}, \quad \overrightarrow{FA}.$$

13. *(Homework)* A plane travels 20km in the direction 30° north of east and then 10 km southeast. Use trigonometry and your calculator to find the final distance and direction of the aircraft from the starting position.

14.* *(Homework)* Let D be the midpoint of the side BC of the triangle ABC.



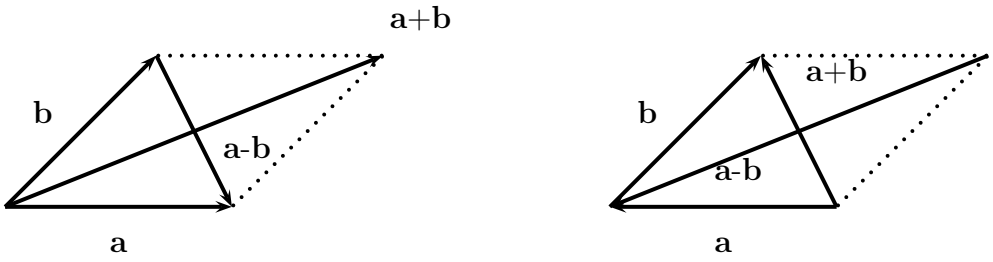
Verify that $\overrightarrow{AD} = \frac{1}{2}\big(\overrightarrow{AB} + \overrightarrow{AC}\big)$.

15.* Prove that the lines joining the midpoints of any quadrilateral form a parallelogram.

Answers to Preparatory Exercises:

1. (i) false (ii) false (iii) true (iv) true

2.



3. (i) $\mathbf{a} - 2\mathbf{b}$ (ii) $\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}$