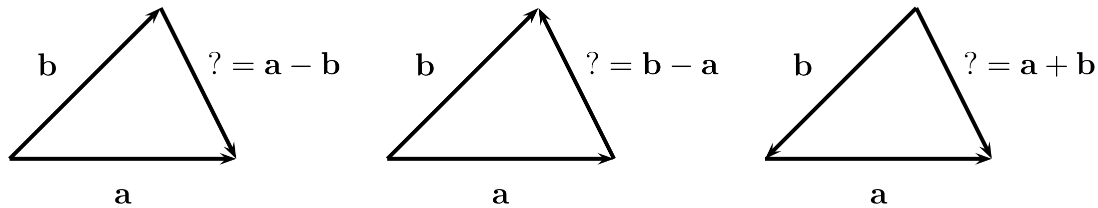
Summer School **Excercise Sheet 1 Solutions** 2012

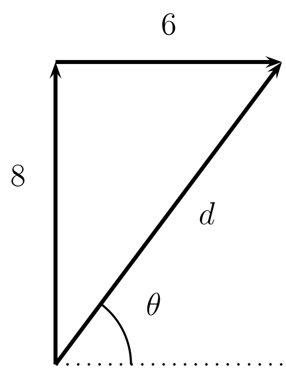
4. (i) 6 (ii) 1 (iii) 6 (iv) 2/3

5.



6. (i) $\mathbf{u}-\mathbf{v}-\mathbf{w}$ (ii) $\mathbf{u}+\mathbf{v}-\mathbf{w}$ (iii) $-\mathbf{u}-\mathbf{v}+\mathbf{w}$

7.



By Pythagoras $d=\sqrt{8^2+6^2}=10$. If θ is the angle to the horizontal then $\cos\theta = 6/10$, yielding an angle $\theta\approx 53°$. Thus the resultant force is 10 newtons in a direction 53° to the horizontal, towards the right.

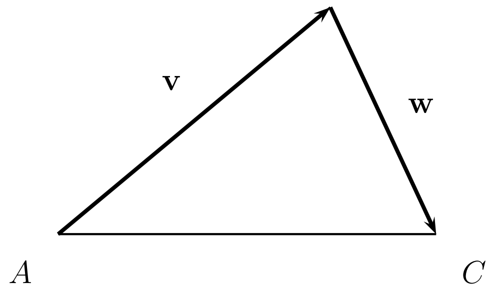
8. The associative law for addition of vectors says that, for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$,

$$\mathbf{u}+(\mathbf{v}+\mathbf{w}) \;=\; (\mathbf{u}+\mathbf{v})+\mathbf{w}\,.$$

To verify this, we suppose that the vectors have been lined up so that the point P is at the tail of $\mathbf{u}$, the point Q is both at the tip of $\mathbf{u}$ and at the tail of $\mathbf{v}$, the point R is both at the tip of $\mathbf{v}$ and at the tail of $\mathbf{w}$, and the point S is at the tip of $\mathbf{w}$. Then

$$\begin{aligned}\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= \overrightarrow{PQ}+\left(\overrightarrow{QR}+\overrightarrow{RS}\right) = \overrightarrow{PQ}+\overrightarrow{QS}\\ &= \overrightarrow{PS}\\ &= \overrightarrow{PR}+\overrightarrow{RS} = \left(\overrightarrow{PQ}+\overrightarrow{QR}\right)+\overrightarrow{RS} = (\mathbf{u}+\mathbf{v})+\mathbf{w}\,.\end{aligned}$$

9.



Place the vectors $\mathbf{v}$ and $\mathbf{w}$ tip-to-tail so that they label two directed edges of a triangle ABC, so that

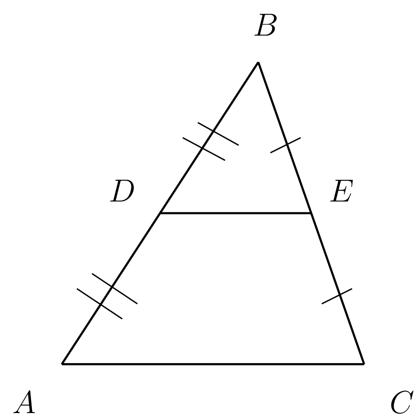
$$\mathbf{v} = \overrightarrow{AB}\,, \qquad \mathbf{w} = \overrightarrow{BC}\,.$$

Then $\mathbf{v} + \mathbf{w} = \overrightarrow{AC}$. The shortest distance between two points is a straight line, so that travelling from A to C via B is at least as far as travelling directly from A to C. Thus

$$|\mathbf{v} + \mathbf{w}| = \left|\overrightarrow{AC}\right| \leq \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right| = |\mathbf{v}| + |\mathbf{w}|\,,$$

which verifies the triangle inequality. This becomes equality precisely when B falls on the direct path joining A to C, so that the triangle becomes degenerate.

10.*



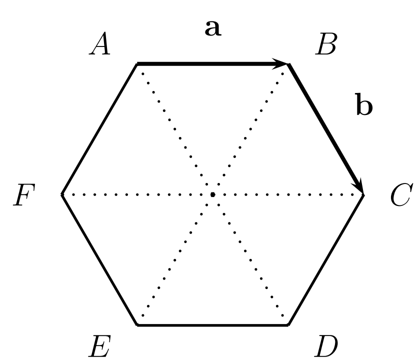
Observe that

$$\overrightarrow{DE} = \overrightarrow{DB} + \overrightarrow{BE} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \frac{1}{2}\overrightarrow{AC}\,.$$

This tells us that the line segment joining D to E is parallel to and half the length of the line segment joining A to C.

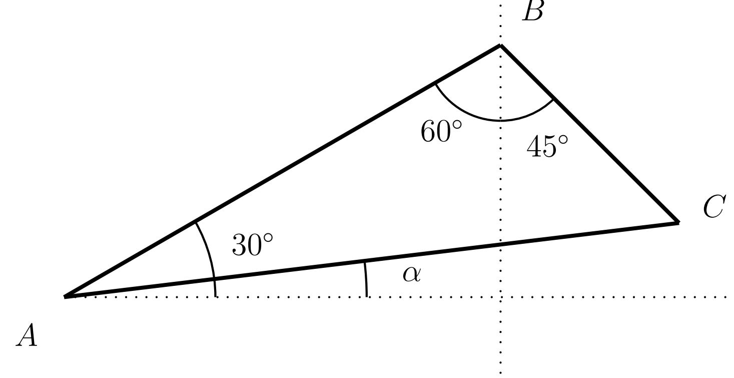
11. $2\mathbf{a} - 3\mathbf{b} = -7\mathbf{u} + 8\mathbf{v}$.

12.



$$\overrightarrow{CD} = \mathbf{b} - \mathbf{a}\,, \quad \overrightarrow{DE} = -\mathbf{a}\,, \quad \overrightarrow{EF} = -\mathbf{b}\,, \quad \overrightarrow{FA} = \mathbf{a} - \mathbf{b}\,.$$

13.



We have $|\overrightarrow{AB}| = 20$ and $|\overrightarrow{BC}| = 10$. By the Cosine Rule,

$$|\overrightarrow{AC}| \;=\; \sqrt{20^2 + 10^2 - 2(10)(20)\cos 105^\circ} \;\approx\; 25\ .$$

By the Sine Rule,

$$\sin(30^\circ - \alpha) \;=\; \frac{10 \sin 105^\circ}{|\overrightarrow{AC}|}\ ,$$

from which it follows that

$$30^\circ - \alpha \;\approx\; 23^\circ\ ,$$

so that $\alpha \approx 7^\circ$. Hence the final distance and direction of the aircraft from the starting point are approximately 25 km and 7° north of east respectively.

14.*

$$\overrightarrow{AD} \;=\; \overrightarrow{AB} + \overrightarrow{BD} \;=\; \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \;=\; \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$$

$$\;=\; \overrightarrow{AB} + \frac{1}{2}(-\overrightarrow{AB} + \overrightarrow{AC}) \;=\; \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} \;=\; \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})\ .$$

15.* Let P, Q, R, S be the respective midpoints of the edges AB, BC, CD, DA of the quadrilateral $ABCD$. Then, by two applications of Exercise 10, firstly to the triangle ABC, and then secondly to the triangle ADC,

$$\overrightarrow{PQ} \;=\; \frac{1}{2}\overrightarrow{AC} \;=\; \overrightarrow{SR}\ ,$$

which is sufficient to prove that $PQRS$ is a parallelogram.