THE UNIVERSITY OF SYDNEY

MATH1002 LINEAR ALGEBRA

Summer School **Exercise Sheet 2** 2012

Preparatory exercises ideally should be attempted before coming to the tutorial. A suggestion is given for exercises to be completed during and after the tutorial. Questions labelled with an asterisk are suitable for students aiming for a credit or higher.

Important Ideas and Useful Facts:

(i) If $\mathbf{v}$ is a geometric vector and λ is a scalar then $|\lambda\mathbf{v}| = |\lambda||\mathbf{v}|$.

(ii) A *unit vector* is a vector of length one. There is always a unit vector pointing in any given direction.

(iii) If λ is a nonzero scalar and $\mathbf{v}$ a vector then we write $\dfrac{\mathbf{v}}{\lambda}$ for the scalar multiple $\dfrac{1}{\lambda}\mathbf{v}$.

(iv) Let $\mathbf{v}$ be a nonzero vector. Then $\widehat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ is called *the hat of* $\mathbf{v}$, or simply $\mathbf{v}$-*hat*, and is the unit vector pointing in the direction of $\mathbf{v}$. It is designed to satisfy the equation

$$\mathbf{v} = |\mathbf{v}|\,\widehat{\mathbf{v}}\,,$$

which captures precisely the idea of $\mathbf{v}$ being characterised by length and direction.

(v) Parallel vectors: Nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if and only if $\mathbf{v} = \lambda\mathbf{w}$ for some nonzero scalar λ.

(vi) Cartesian form: The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ point in the directions of the positive x, y and z-axes respectively. If $\mathbf{v}$ is the position vector of the point $P(a,b,c)$ then

$$\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}\,,$$

called the *Cartesian form* of $\mathbf{v}$. In this case, a, b, c are called the *components* of $\mathbf{v}$.

(vii) Component-wise operations: To add, subtract or negate vectors, simply add, subtract or negate respective components. To multiply a vector by a scalar, simply multiply the components by the scalar.

(viii) If $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ then $|\mathbf{v}| = \sqrt{a^2+b^2+c^2}$.

(ix) If $P(a_1,b_1,c_1)$ and $Q(a_2,b_2,c_2)$ are points in space then

$$\overrightarrow{PQ} = (a_2-a_1)\mathbf{i} + (b_2-b_1)\mathbf{j} + (c_2-c_1)\mathbf{k}\,.$$

(x) Linear independence of two vectors: Two vectors $\mathbf{v}$ and $\mathbf{w}$ are *linearly independent* if they are not parallel, which is equivalent to saying that the vector equation

$$a\,\mathbf{v} + b\,\mathbf{w} = \mathbf{0}$$

implies that the scalars a and b must both be zero.

Preparatory Exercises (answers below):

1. In the xy-plane the z-coordinate is ignored and $\mathbf{i}$ and $\mathbf{j}$ are, as usual, unit vectors in the positive x and y-directions respectively. Let P be the point $(3, 1)$ and Q the point $(4, -2)$ in the xy-plane. As usual the origin $(0, 0)$ is denoted by O.

(i) Write down the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(ii) Write down the displacement vector $\overrightarrow{PQ}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(iii) Write down the coordinates of the point R such that $\overrightarrow{OR} = \overrightarrow{PQ}$.

2. Let P be the point $(3, 1, -2)$ and Q the point $(4, -2, 5)$ in space. As usual the origin $(0, 0, 0)$ is denoted by O.

(i) Write down the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

(ii) Write down the displacement vector $\overrightarrow{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

(iii) Write down the coordinates of the point R such that $\overrightarrow{OR} = \overrightarrow{PQ}$.

3. Given that

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,, \qquad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}\,, \qquad \mathbf{c} = 3\mathbf{i} - 4\mathbf{k}\,,$$

find

(i) $\mathbf{a} + \mathbf{b}$ (ii) $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c}$ (iii) $|\mathbf{a}|$ (iv) $\widehat{\mathbf{a}}$ (v) $\widehat{\mathbf{c}}$

Tutorial Exercises:

4. Given points $A(-2, 3)$ and $B(4, -1)$ in the xy-plane, find

(i) the position vectors of A and B in terms of $\mathbf{i}$ and $\mathbf{j}$;

(ii) the displacement vector $\overrightarrow{AB}$ in terms of $\mathbf{i}$ and $\mathbf{j}$;

(iii) the unit vector pointing from A towards B;

(iv) the unit vector pointing from B towards A.

5. Given that $\mathbf{a} = \sqrt{3}\,\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \dfrac{1}{\sqrt{2}}\left(\mathbf{i} - \mathbf{j}\right)$, find

(i) $|\mathbf{a}|$ (ii) $|\mathbf{b}|$ (iii) $\widehat{\mathbf{a}}$ (iv) $\widehat{\mathbf{b}}$ (v) $2\sqrt{3}\,\widehat{\mathbf{a}} + \sqrt{2}\,\widehat{\mathbf{b}}$

6. Given points $A(4, -1, 5)$ and $B(6, -1, -2)$ in space, find

(i) the position vectors of A and B in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;

(ii) the displacement vector $\overrightarrow{AB}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;

(iii) the unit vector pointing from A towards B;

(iv) the unit vector pointing from B towards A.

7. Let $\mathbf{v} = 2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$ and $\mathbf{w} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Find

(i) $-\mathbf{v}$ (ii) $\mathbf{w} - \mathbf{v}$ (iii) $2\mathbf{v}$ (iv) $3\mathbf{w}$ (v) $2\mathbf{v} - 3\mathbf{w}$
(vi) $|\mathbf{v}|$ (vii) $|\mathbf{w}|$ (viii) $\widehat{\mathbf{v}}$ (ix) $\widehat{\mathbf{w}}$ (x) $|\mathbf{v} + \mathbf{w}|$

8. (suitable for group discussion) Let $\mathbf{i}$ and $\mathbf{j}$ denote displacements of 1 km east and north respectively. An aeroplane travels 300km southeast and then 150 km in the direction 30° west of north. Find

(i) the above displacements of the aeroplane and their vector sum in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$;
(ii) the final distance (to the nearest km) and direction (to nearest degree, south of east) of the aeroplane from the starting position.

9. (suitable for group discussion) Let $ABCDEF$ be a regular hexagon. True or false:

(i) $\overrightarrow{AC} = \overrightarrow{FD}$ (ii) $\overrightarrow{AC} = \overrightarrow{DF}$ (iii) $\overrightarrow{AC} = \overrightarrow{BD}$ (iv) $|\overrightarrow{AC}| = |\overrightarrow{BD}|$
(v) $|\overrightarrow{AC}| = |\overrightarrow{AD}|$ (vi) The line segments AD and BE bisect each other.

10.* Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors which are not parallel (so are linearly independent) and the following vector equation holds for some scalars α and β:

$$\mathbf{v} + \alpha(\mathbf{w} - \mathbf{v}) \;=\; \beta\big(\mathbf{v} + \frac{1}{2}\mathbf{w}\big)\ .$$

Find α and β.

Further Exercises:

11. Consider the points $A(1, 2, -3)$, $B(-2, 1, 1)$ and $C(0, 2, 1)$.

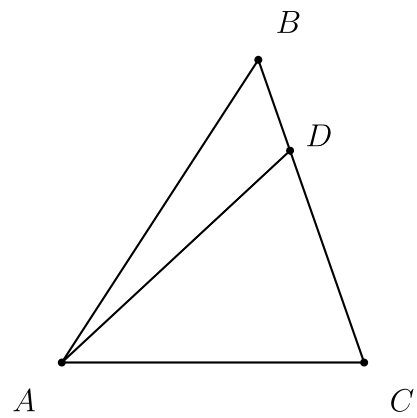
(i) Find the point D such that $ABCD$ is a parallelogram.
(ii) Let P be the midpoint of AC. Find the Cartesian form of $\overrightarrow{OP}$.
(iii) Find the Cartesian forms of $\overrightarrow{BP}$ and $\overrightarrow{PD}$, and deduce that the diagonals AC and BD bisect each other.
(iv) Find the lengths of $\overrightarrow{AC}$ and $\overrightarrow{BD}$. Is the parallelogram $ABCD$ a rectangle?

12. Let $\mathbf{v} = \overrightarrow{PQ}$ where $P = (-3, 2, 0)$ and $Q = (4, -2, 3)$. Find the Cartesian form of $\mathbf{v}$, the length of $\mathbf{v}$ and the angles $\mathbf{v}$ makes (to the nearest degree) with each of the positive x, y and z-axes. (The cosine of each angle will be the relevant component divided by the length of the vector.)

13. Find the scalars α, β and γ such that

(i) $3\mathbf{i} + \mathbf{j}$ is parallel to $\alpha\mathbf{i} - 4\mathbf{j}$ (ii) $3\mathbf{i} + \beta\big(\mathbf{j} - \mathbf{k}\big)$ is parallel to $\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$
(iii) $3\mathbf{i} + \gamma\big(\mathbf{j} + 3\mathbf{k}\big)$ has the same length as $12\mathbf{i} - 5\mathbf{k}$.

14.* Let D be the point which divides the side BC of the triangle ABC in the ratio $\alpha : \beta$.



Verify that $\overrightarrow{AD} = \dfrac{\beta\,\overrightarrow{AB} + \alpha\,\overrightarrow{AC}}{\alpha + \beta}$.

15.* The line joining the vertex of a parallelogram to the midpoint of an opposite side divides one of the diagonals into two pieces of unequal lengths. Find the ratio of the lengths of these pieces.

[Hints: (1) check your answer directly after conjecturing (say from a diagram) the value of the ratio; or (2) use Exercise 10 (guaranteed to work but takes longer), exploiting the fact that adjacent sides of the parallelogram are not parallel.]

Answers to Preparatory Exercises:

1. (i) $\overrightarrow{OP} = 3\mathbf{i} + \mathbf{j}$, $\overrightarrow{OQ} = 4\mathbf{i} - 2\mathbf{j}$, (ii) $\overrightarrow{PQ} = \mathbf{i} - 3\mathbf{j}$ (iii) $R = (1, -3)$

2. (i) $\overrightarrow{OP} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\overrightarrow{OQ} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, (ii) $\overrightarrow{PQ} = \mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$

(iii) $R = (1, -3, 7)$

3. (i) $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + \mathbf{k}$ (ii) $\mathbf{a} + 3\mathbf{b} - 2\mathbf{c} = -\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$ (iii) $|\mathbf{a}| = 3$

(iv) $\widehat{\mathbf{a}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ (v) $\widehat{\mathbf{c}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k}$