Summer School **Excercise Sheet 2 Solutions** 2012

4. (i) $\overrightarrow{OA} = -2\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OB} = 4\mathbf{i} - \mathbf{j}$. (ii) $\overrightarrow{AB} = 6\mathbf{i} - 4\mathbf{j} = 2(3\mathbf{i} - 2\mathbf{j})$.

(iii) the unit vector pointing from A towards B is $\dfrac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$.

(iv) the unit vector pointing from B towards A is $-\dfrac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$.

5. (i) $|\mathbf{a}| = 2$. (ii) $|\mathbf{b}| = 1$. (iii) $\widehat{\mathbf{a}} = \dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

(iv) $\widehat{\mathbf{b}} = \dfrac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ (v) $2\sqrt{3}\,\widehat{\mathbf{a}} + \sqrt{2}\,\widehat{\mathbf{b}} = 4\,\mathbf{i} + (\sqrt{3} - 1)\,\mathbf{j}$

6. (i) $\overrightarrow{OA} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\overrightarrow{OB} = 6\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. (ii) $\overrightarrow{AB} = 2\mathbf{i} - 7\mathbf{k}$.

(iii) the unit vector pointing from A towards B is $\dfrac{1}{\sqrt{53}}(2\mathbf{i} - 7\mathbf{k})$.

(iv) the unit vector pointing from B towards A is $-\dfrac{1}{\sqrt{53}}(2\mathbf{i} - 7\mathbf{k})$.

7. (i) $-\mathbf{v} = -2\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}$ (ii) $\mathbf{w} - \mathbf{v} = 2\mathbf{i} + 8\mathbf{j} - 13\mathbf{k}$

(iii) $2\mathbf{v} = 4\mathbf{i} - 12\mathbf{j} + 18\mathbf{k}$ (iv) $3\mathbf{w} = 12\mathbf{i} + 6\mathbf{j} - 12\mathbf{k}$

(v) $2\mathbf{v} - 3\mathbf{w} = -8\mathbf{i} - 18\mathbf{j} + 30\mathbf{k}$ (vi) $|\mathbf{v}| = 11$ (vii) $|\mathbf{w}| = 6$

(viii) $\hat{\mathbf{v}} = \frac{1}{11}(2\mathbf{i} - 6\mathbf{j} + 9\mathbf{k})$ (ix) $\hat{\mathbf{w}} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ (x) $|\mathbf{v} + \mathbf{w}| = \sqrt{77}$

8. (i) The displacement 300 km southeast is represented by the vector $150\sqrt{2}\,(\mathbf{i} - \mathbf{j})$ and 150 km 30° west of north by the vector $75\,(-\mathbf{i} + \sqrt{3}\,\mathbf{j})$. The net displacement is represented by

$$(150\sqrt{2} - 75)\,\mathbf{i} + (75\sqrt{3} - 150\sqrt{2})\,\mathbf{j}\ .$$

(ii) The final distance from the starting position is

$$\sqrt{(150\sqrt{2} - 75)^2 + (75\sqrt{3} - 150\sqrt{2})^2} \;\approx\; 160 \quad \text{km}\ .$$

The tangent of the angle south of east is $\dfrac{150\sqrt{2} - 75\sqrt{3}}{150\sqrt{2} - 75}$ yielding an angle of approximately 31°.

9. (i) True (ii) False (iii) False (iv) True (v) False (vi) True

10.* Rearranging the equation gives

$$(1-\alpha-\beta)\,\mathbf{v} + \left(\alpha - \frac{\beta}{2}\right)\mathbf{w} \;=\; \mathbf{0}\,,$$

so that, by linear independence, $1-\alpha-\beta \;=\; 0 \;=\; \alpha - \dfrac{\beta}{2}$. Solving simultaneously yields $\alpha = 1/3$, $\beta = 2/3$.

11. (i) We want $D(x,y,z)$ such that $\overrightarrow{AB} = \overrightarrow{DC}$, so that

$$-3\mathbf{i} - \mathbf{j} + 4\mathbf{k} \;=\; -x\,\mathbf{i} + (2-y)\mathbf{j} + (1-z)\mathbf{k}\,,$$

yielding $x=3$, $y=3$, $z=-3$. Hence $D=(3,3,-3)$.

(ii) The coordinates of P are the averages of the respective coordinates of A and C, so $P = (\frac{1}{2}, 2, -1)$ and $\overrightarrow{OP} = \frac{1}{2}\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

(iii) We have $\overrightarrow{BP} = \overrightarrow{PD} = \frac{5}{2}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, so that P must be the midpoint of the line segment joining B and D. Thus the diagonals AC and BD bisect each other.

(iv) We have

$$\left|\overrightarrow{AC}\right| \;=\; |-\mathbf{i} + 4\mathbf{k}| \;=\; \sqrt{17}\,, \qquad \left|\overrightarrow{BD}\right| \;=\; |5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}| \;=\; 3\sqrt{5}\,.$$

Since these lengths are different, the parallelogram $ABCD$ a not a rectangle.

12. We have

$$\mathbf{v} \;=\; 7\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}\,, \qquad |\mathbf{v}| \;=\; \sqrt{74}\,,$$

so the cosines of the angles made with the x, y and z-axes are

$$\frac{7}{\sqrt{74}}\,, \quad -\frac{4}{\sqrt{74}}\,, \quad \frac{3}{\sqrt{74}}\,,$$

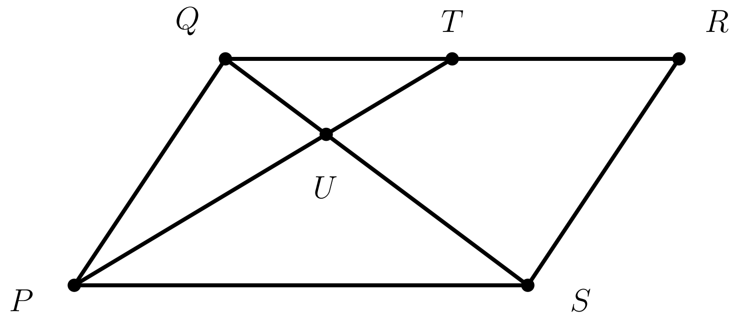
yielding angles of approximately 36°, 118° and 70° respectively.

13. (i) $\alpha = -12$ (ii) $\beta = -12$ (iii) $\gamma = \pm 4$

14.* Observe that

$$\begin{aligned}
\overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD} \;=\; \overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}\overrightarrow{BC} \;=\; \overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}\big(\overrightarrow{BA} + \overrightarrow{AC}\big) \\
&= \overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}\big(-\overrightarrow{AB} + \overrightarrow{AC}\big) \;=\; \left(1 - \frac{\alpha}{\alpha+\beta}\right)\overrightarrow{AB} + \frac{\alpha}{\alpha+\beta}\overrightarrow{AC} \\
&= \frac{\beta\,\overrightarrow{AB} + \alpha\,\overrightarrow{AC}}{\alpha+\beta}\,.
\end{aligned}$$

15.* Consider the following parallelogram $PQRS$, and let U be the point of intersection of PT with QS, where T is the midpoint of QR.



Then, for some scalars α and β,

$$\overrightarrow{QU} = \alpha\overrightarrow{QS}, \qquad \overrightarrow{PU} = \beta\overrightarrow{PT}.$$

Put

$$\mathbf{v} = \overrightarrow{PQ}, \qquad \mathbf{w} = \overrightarrow{PS}.$$

On the one hand,

$$\overrightarrow{PU} = \overrightarrow{PQ} + \overrightarrow{QU} = \mathbf{v} + \alpha\overrightarrow{QS} = \mathbf{v} + \alpha\left(\overrightarrow{QP} + \overrightarrow{PS}\right) = \mathbf{v} + \alpha\left(\mathbf{w} - \mathbf{v}\right),$$

whilst, on the other hand,

$$\overrightarrow{PU} = \beta\overrightarrow{PT} = \beta\left(\overrightarrow{PQ} + \overrightarrow{QT}\right) = \beta\left(\mathbf{v} + \frac{1}{2}\overrightarrow{QR}\right) = \beta\left(\mathbf{v} + \frac{1}{2}\mathbf{w}\right),$$

whence

$$\mathbf{v} + \alpha\left(\mathbf{w} - \mathbf{v}\right) = \beta\left(\mathbf{v} + \frac{1}{2}\mathbf{w}\right).$$

By the calculation in Exercise 10,

$$\alpha = \frac{1}{3}, \qquad \beta = \frac{2}{3}.$$

Hence the ratio of the length of QU to the length of US is $1:2$.

An alternative (and faster) solution is to conjecture that the ratio is $1:2$ and simply check that

$$\overrightarrow{PQ} + \frac{1}{3}\overrightarrow{QS} = \overrightarrow{PQ} + \frac{1}{3}\left(\overrightarrow{QR} + \overrightarrow{RS}\right) = \overrightarrow{PQ} + \frac{2}{3}\overrightarrow{QT} - \frac{1}{3}\overrightarrow{PQ} = \frac{2}{3}\left(\overrightarrow{PQ} + \overrightarrow{QT}\right) = \frac{2}{3}\overrightarrow{PT},$$

which confirms that PT intesects QS one third of the way from Q to S.